A New Similarity Measure for Complex Valued Data

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Abstract: As possible basis for a holographic Perceptual Quality Predictor, a new model of "similarity" is proposed. Defined as a product of magnitude error, relative- and absolute phase error, it allows for local and global interpretation.

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1. Introduction

Quantitative rating of perceptual fidelity of reconstructed objects from compressed holograms, complex wavefield recordings, is highly desired. The complexity of the Human Visual System regarding depth perception and the lack of standard frameworks to perform such evaluations increases the difficulty of such a task. An efficient holographic Perceptual Quality Predictor (PQP) algorithm should be able to predict the perceptual quality of the reconstructed object only by analyzing the hologram. To do so, a first step will be to accurately measure the the amount of degradation in a compressed hologram compared to its uncompressed reference.

Since the absolute value of a complex number is equal to the sum of the real and imaginary part, constructions over $\mathbb{R}^2$ such as the Mean Squared Error (MSE), SSIM [1], or measurement of enhancement by entropy the EME [2], have been used to quantify dissimilarity. Measures like the CW-SSIM [3, 4], are generalizations of notions on $\mathbb{R}^2$ but have not been natively crafted for complex values. As already mentioned in [5], a more versatile set of tools is desired and future PQP algorithms may require other ways for error quantification.

For example the MSE, measuring the energy of degradation, is not bounded and hard to interpret locally - especially for complex values. Boundedness allows to introduce the notions of: "similarity", and local as well as global interpretability, which are essential notions for PQPs.

We present a new model for a similarity measure on the complex domain that addresses the mentioned shortcomings. The model is detailed in sec. 2, first numerical experiments are presented in sec. 3 and we conclude with a brief conclusion and outlook in sec. 4.

2. Method

Our new model for a similarity measure on complex numbers is primarily designed to have an easy local and global interpretation. It might be used as an alternative to the MSE. Its secondary design goals were boundedness to $(0, 1)$, difference weighting relative to "magnitudes" of the signal, steerable similarity between original and negative. symmetry with respect to ordering of the arguments and the regularity of a continuous function or better.

For the sake of simplicity we will focus our explanation in this section on comparing the values of just two complex numbers. Meaningful generalizations to complex valued ND arrays depend on the usage scenario. In case a simple absolute error measure is required, the arithmetic mean or the median over all single pixel scores suffice. More sophisticated weighted means will be motivated by the PQP.

The model defines similarity, $\rho$ close to or equal 1, as the product of three factors measuring: the difference in magnitude; the relative phase error; and the absolute phase error. This product structure allows to perform an easy local analysis. $\rho \approx 0$ if and only if at least one component fails. Components can be adapted to any dynamic range of
the data via parameters or swapped out completely.

\[
\rho(z_1, z_2) := \frac{1}{(1 + d) \exp \left( -a \angle (z_1, z_2)^2 \right) - d} \left( \lambda + (1 - \lambda) H \left( \frac{\pi}{2} - \angle (z_1, z_2) \right) \right)
\]

\[
a := \frac{4}{\pi^2} \ln \left( \frac{1 + d}{d} \right)
\]

\[
\angle (z_1, z_2) := \begin{cases} 
\left| \text{ang} (z_1) - \text{ang} (z_2) \right|, & < \pi \\
\left| 2\pi - \text{ang} (z_1) - \text{ang} (z_2) \right|, & \text{otherwise}
\end{cases}
\]

with absolute complex phase, \( \text{ang}(z) \in [0, 2\pi) \), \( \rho \in [1, \infty) \), \( d \in (0, 1) \), \( \lambda \in [0, 1] \). For practical purposes and to preserve regularity, the discontinuity of \((M)\) in \((0, 0)\) was smoothened out by:

\[
f(z_1, z_2) := \begin{cases} 
\max \left\{ (M), 1 - \frac{\sqrt{|z_1|^2 + |z_2|^2}}{\varepsilon} \right\}, & \sqrt{|z_1|^2 + |z_2|^2} \leq \varepsilon \\
(M), & \text{otherwise}
\end{cases}
\]

with \( \varepsilon \ll 1 \). It lower bounds the first factor by a cone around \((0, 0)\) of radius \( \varepsilon \) that effectively is to be interpreted as the minimal similarity in magnitude of two values located within the cone support.

One exemplary combination leading to a pleasing result is shown in figure 1. We used \( p = 2, d = 10^{-4}, \lambda = 1/2, \) and \( \varepsilon = 10^{-8} \).

![Figure 1](image.png)

Figure 1. One example of the new similarity measure shows how similarity of complex numbers can be defined. The smaller peak is designed to reflect the limited similarity of the original and its negative. The black ball indicates the reference number at height 1.

The magnitude error extends the idea of measuring relative magnitude errors by allowing powers \( p \neq 1 \). The effect of a constant error will decrease as the scale of the numbers increases.

The phase error may be split up into two parts to discriminate negatives from originals. The relative phase error, will measure the phase difference directly and the absolute phase error, will distinguish originals from their negatives.

The relative phase error is \( \pi \)-periodic, starting its period at \(-\pi/2\) and rates any two complex numbers 0, if their phase difference is \( \pi/2 \). This corresponds to the physical interpretation of the phase in linearly polarized electromagnetic fields. We based this error measure on a periodic continuation of a modified Gaussian distribution defined on \([-\pi/2, \pi/2]\). It takes a parameter \( d \) that is used to lower and rescale the Gaussian distribution such that it is 1 in 0 and 0 in \( \{\pm \pi/2\} \). It effectively controls the standard deviation. A mean value \( \neq 0 \) is not meaningful. This component is \( C^\infty \) everywhere.

The absolute phase error, measured with a mirrored and shifted Heaviside function has takes a height parameter \( \lambda \), a distinction between the two half-planes \( \{ z \in \mathbb{C} \mid \angle(z, w) \leq \pi/2 \} \) and \( \{ z \in \mathbb{C} \mid \angle(z, w) > \pi/2 \} \) for any given \( w \in \mathbb{C} \setminus \{0\} \) is made. Its jump discontinuities are hidden through the zeros of the relative phase error and thus the whole measure \( \rho \) is still at least a continuous function.

If the overall phase error is kept at a fixed level, one can see from the choice of the relative magnitude error measure, that the region of high similarity \( (\rho \approx 1) \) in \( \mathbb{C} \) is contained in a cone and being centered around the direction of \( z \). Depending on \( \lambda \) it can also become a double cone, with one cone being centered in \( z \) direction.
3. Numerical experiments

We selected three $8192 \times 8192$ computer generated holograms namely Ball, Cat and Chess from [6]. Their 8bit-quantized real and imaginary parts were separately compressed by the JPEG2000 encoder with default settings. We experimented with 5 different bit-rates per real and imaginary channel. The trends of MSE, measuring the error, and the median of $\rho$ with $p = 4$, $\lambda = 1/2$, $d = 10^{-4}$, measuring the $1 – \rho$ error, match up closely, see figure 2, and thus $\rho$ could be used as a local and global quality indicator.

![Figure 2](image.png)

**Figure 2.** Rate-Distortion analysis for 3 compressed CGHs in terms of MSE and the proposed model. The standad JPEG2000 encoder were used with bit-rates: 0.0625, 0.125, 0.25, 0.5 and 1 bits per pixel (bpp) per real and imaginary channel of the complex wavefields.

4. Conclusion & future directions

A new complex similarity model is proposed in order to address the shortcomings of the available complex error measures in particular MSE. The model may not only be utilized as a stand-alone complex-signal similarity measure but paves also the way toward designing efficient PQP algorithms for complex valued data like digital holograms.

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