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FROBENIUS AND SEPARABLE FUNCTORS FOR THE CATEGORY OF ENTWINED MODULES OVER COWREATHS, II: APPLICATIONS

D. BULACU, S. CAENEPEEL, AND B. TORRECILLAS

ABSTRACT. Let H be a quasi-Hopf algebra. We apply results obtained in [8] to give necessary and sufficient conditions for the forgetful functor from Doi-Hopf modules, two-sided Hopf modules or Yetter-Drinfeld modules over H to representations of the underlying algebra to be Frobenius (resp. separable). We show that in some situations these conditions reduce to the unimodularity and/or (co)semisimplicity of the quasi-Hopf algebra H .

INTRODUCTION

This paper is the final one in a series that had as main goal the study of Frobenius and separable properties for forgetful functors defined on categories of entwined modules over cowreaths obtained from certain quasi-Hopf actions and coactions. We initiated this study in [6] where we presented the connection between (co)wreaths and certain (co)ring structures, as well as their connection with certain entwined modules. This led us to a connection between Frobenius/separable wreaths in 2-categories and algebra extensions produced by them that are Frobenius/separable, see [11]. We should note that the general theory of Frobenius monads was considered by Street in [24]. Last but not least, in [8] we developed a theory that allows us not only to achieve our mentioned goal but also to produce similar results for other generalizations of Hopf algebras.

Frobenius/separable functors are strictly related to Frobenius/separable algebra extensions, see [11, 14]. From this perspective, in Hopf algebra theory, the study of Frobenius and separability for Doi-Hopf modules was done in [13, 15, 16]. Afterwards this study was refined and applied to entwined modules by Brzeziński in [2]. In either case the Frobenius or separability property reduces to a certain morphism to be an isomorphism or to a list of conditions imposed to a certain bilinear form, respectively. In [8] we uncovered a monoidal interpretation for these results. Namely, Doi-Hopf modules can be regarded as entwined modules over a cowreath (or mixed wreath [25] or generalized entwined structure), and the latter is nothing but a coalgebra in the Eilenberg-Moore category $\mathcal{T}_A^\#$ associated to a certain algebra A ; then the forgetful functor is Frobenius/separable if and only if this coalgebra is Frobenius/coseparable within the monoidal category $\mathcal{T}_A^\#$. We should point out that these characterizations apply to any cowreath in a monoidal category for which the unit object is what we called in [11] a left \otimes -generator, a property stronger than the generator property. So they apply to the category of vector spaces, the category of bimodules ${}_R\mathcal{M}_R$ over an Azumaya k -algebra R , the category of finite dimensional Hilbert complex vector spaces FdHilb or to the category \mathcal{Z}_k as introduced in [12]. For more details we refer to [11, Examples 3.2].

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When we pass to quasi-Hopf algebras, the Doi-Hopf module categories cannot be viewed as entwined modules over entwining structures. This motivated us to investigate monoidal cowreaths (which can be regarded as generalized entwining structures), as they appear naturally from the applications that we have in mind: the study of categories of Doi-Hopf modules, two-sided Hopf modules and Yetter-Drinfeld modules over a quasi-Hopf algebra. Using the general theory performed in [8], we find necessary and sufficient conditions for which they are Frobenius or separable cowreaths but these conditions are quite technical even in the Hopf algebra case. Nevertheless, with some effort we were able to find concrete examples, and also to rephrase in some cases the Frobenius/separability property of the cowreath (or, equivalently, of the forgetful functor to the representations of the underlying algebra) in terms of unimodularity or/and (co)semisimplicity of H . We should also stress the fact that we invented techniques that allow to produce in each case that we are dealing with results that are new even for Hopf algebras; see for instance the results related to the Frobenius/separability properties for the category of two-sided Hopf modules as well as the results related to the separability property for Yetter-Drinfeld modules.

The paper is organized as follows. In Section 1, we present preliminary results on Frobenius/separable cowreaths in monoidal categories and quasi-Hopf algebras. In Section 2, we study the Frobenius/separable property for the categories of Doi-Hopf modules over a quasi-Hopf algebra H , as introduced in [4]. Let A be a right H -comodule algebra. To a coalgebra C in the monoidal category \mathcal{M}_H , we can associate a coalgebra C in $\mathcal{T}_A^\#$. If C is a Frobenius coalgebra in \mathcal{M}_H , then the forgetful functor from the category of Doi-Hopf modules to the category of A -modules is Frobenius, see Proposition 2.3. The converse property holds if $A = H$, see Theorem 2.4. We have a similar result on the separability of the forgetful functor, see Proposition 2.6. Moreover, the relationship between coseparability of C as a coalgebra in \mathcal{M}_H and in $\mathcal{T}_A^\#$ is well-understood: there is a bijection between normalized Casimir morphisms for C in \mathcal{M}_H and normalized Casimir morphisms for C in $\mathcal{T}_A^\#$ that satisfy the additional condition (2.18), see Theorem 2.7. Consequently, if the functor is separable then this does not imply $C \in \mathcal{M}_H$ coseparable; as we will see such a situation occurs in the case when we work with two-sided Hopf modules, a particular class of Doi-Hopf modules.

In Section 3 we focus on the category of two-sided Hopf modules, see [4]. Since this category is isomorphic to a suitable category of Doi-Hopf modules, we can apply the results of Section 2. Some interesting results can be obtained in the situation where $A = C = H$. The forgetful functor $F : {}_H\mathcal{M}_H^H \rightarrow \mathcal{M}_H$ is separable if and only if H is unimodular, see Theorem 3.7. H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple, see Proposition 3.6. Thus the coseparability of the coalgebra H in ${}_H\mathcal{M}_H$ is not equivalent to the separability of the functor F , as it happens in the Frobenius case: F is Frobenius if and only if F is separable, if and only if H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, i.e. H is unimodular.

The Frobenius/separability property for the forgetful functor defined on a category of Yetter-Drinfeld modules (see [9]) is studied in Section 4. The category of Yetter-Drinfeld modules is also isomorphic to a suitable category of Doi-Hopf modules, so that the results of Section 2 can be applied, see Proposition 4.2. We obtain complicated conditions, but better results can be obtained in the situation where $A = C = H$, that is, when we deal with classical Yetter-Drinfeld modules. The first main result is Theorem 4.5, telling that $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is Frobenius if and only if H is finite dimensional and unimodular, if and only if H is finite dimensional and Frobenius as a coalgebra in ${}_H\mathcal{M}_H$. The second main result is Theorem 4.9, saying that $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is separable if and only if H is a coseparable coalgebra in ${}_H\mathcal{M}_H$, i.e. H is unimodular and cosemisimple. If H is finite dimensional, then we can consider the Drinfeld double $D(H)$, and the algebra extension $H \hookrightarrow D(H)$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple). Note that the proofs of these results are based on the structure theorem for two-sided Hopf H -bimodules, see [18]. Another characterization about the separability of $D(H)$ over H , in terms of so-called ad-(co)invariant integrals, was obtained by Ardizzoni in [1, Section 3].

1. PRELIMINARIES

1.1. The categories \mathcal{T}_A and $\mathcal{T}_A^\#$. For the definition of a monoidal category $(\mathcal{C}, \otimes, 1, a, l, r)$ and related topics we refer for instance to [19, 20]. We will often delete the tensor symbol \otimes , and write $X \otimes Y = XY$, provided that X, Y are objects of \mathcal{C} . We write X^n for the tensor product of n copies of X . Furthermore, the identity morphism of an object $X \in \mathcal{C}$ will be denoted by Id_X or simply X . Let $(A, \underline{m}_A, \underline{\eta}_A)$ be an algebra in \mathcal{C} . A (right) transfer morphism (called transition map by Tambara [26]) through A is a pair (X, ψ) , with $X \in \mathcal{C}$ and $\psi : XA \rightarrow AX$ in \mathcal{C} such that

$$(1.1) \quad (a) \ \psi \circ X\underline{m}_A = \underline{m}_A X \circ A\psi \circ \psi A \text{ and } (b) \ \psi \circ X\underline{\eta}_A = \underline{\eta}_A X.$$

The categories \mathcal{T}_A and $\mathcal{T}_A^\#$ coincide at the level of objects; their objects are right transfer morphisms through A . A morphism $X \rightarrow Y$ in \mathcal{T}_A is a morphism $f : X \rightarrow Y$ in \mathcal{C} such that $\psi \circ fA = Af \circ \psi$. A morphism $X \rightarrow Y$ in $\mathcal{T}_A^\#$ is a morphism $f : X \rightarrow AY$ in \mathcal{C} such that

$$(1.2) \quad \underline{m}_A Y \circ Af \circ \psi = \underline{m}_A Y \circ A\psi \circ fA.$$

The composition of two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in $\mathcal{T}_A^\#$ is $g \bullet f = \underline{m}_A Z \circ Ag \circ f$. The identity on (X, ψ) is $\underline{\eta}_A X$. The tensor product of X and Y is $XY = (XY, \psi_X \cdot \psi_Y = \psi_X Y \circ X\psi_Y)$. The tensor product of $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in $\mathcal{T}_A^\#$ is given by the composition $\underline{m}_A XY \circ A\psi Y \circ fg$. The unit object is $(1, A)$. \mathcal{T}_A and $\mathcal{T}_A^\#$ are strict monoidal categories, and we have a strong monoidal functor $F : \mathcal{T}_A \rightarrow \mathcal{T}_A^\#$, which is the identity on objects, and $F(f) = \underline{\eta}_A f$, for $f : X \rightarrow Y$ in \mathcal{T}_A . If a morphism in $\mathcal{T}_A^\#$ is of the form $\underline{\eta}_A f$, with $f : X \rightarrow Y$ in \mathcal{C} , then f is a morphism in \mathcal{T}_A .

We can recall now from [8] the notion of cowreath and of entwined module over a cowreath.

Definition 1.1. A cowreath (mixed wreath or generalized entwining structure) in \mathcal{C} is a triple (A, X, ψ) , where A is an algebra in \mathcal{C} , and (X, ψ) is a coalgebra in $\mathcal{T}_A^\#$, i.e. it is an object $(X, \psi) \in \mathcal{T}_A^\#$ together with morphisms $\delta : X \rightarrow AX^2$ and $\epsilon : X \rightarrow A$ in \mathcal{C} such that the following relations hold:

$$(1.3) \quad \begin{aligned} (a) \quad & \underline{m}_A X^2 \circ A\psi X \circ AX\psi \circ \delta A = \underline{m}_A X^2 \circ A\delta \circ \psi, \\ (b) \quad & \underline{m}_A X^3 \circ A\delta X \circ \delta = \underline{m}_A X^3 \circ A\psi X^2 \circ AX\delta \circ \delta, \\ (c) \quad & \underline{m}_A \circ A\epsilon \circ \psi = \underline{m}_A \circ \epsilon A, \\ (d) \quad & \underline{m}_A X \circ A\epsilon X \circ \delta = \underline{\eta}_A X, \\ (e) \quad & \underline{m}_A X \circ A\psi \circ AX\epsilon \circ \delta = \underline{\eta}_A X. \end{aligned}$$

Conditions (a) and (c) mean that δ and ϵ define morphisms $X \rightarrow X^2$ and $X \rightarrow 1$ in $\mathcal{T}_A^\#$. (b) is the coassociativity of the comultiplication δ and (d) and (e) are the left and right counit property.

Definition 1.2. An entwined module over a cowreath (A, X, ψ) consists of a right A -module M in \mathcal{C} and a right A -linear morphism $\rho : M \rightarrow MX$ in \mathcal{C} satisfying

$$(1.4) \quad \rho X \circ \rho = \mu X^2 \circ M\delta \circ \rho;$$

$$(1.5) \quad \mu \circ M\epsilon \circ \rho = M,$$

where $\mu : M \otimes A \rightarrow M$ is the morphisms in \mathcal{C} that defines the right A -module structure on M .

(1.4) is the coassociativity of the coaction ρ , and (1.5) is the counit property. The fact that ρ is right A -linear is expressed by the formula

$$(1.6) \quad \rho \circ \mu = \mu X \circ M\psi \circ \rho A.$$

We denote by $\mathcal{C}(\psi)_A^X$ the category of (right) entwined modules over the cowreath (A, X, ψ) in the monoidal category \mathcal{C} . The morphisms are right A -linear morphisms in \mathcal{C} that behaves well with respect to the right X -coactions.

1.2. Frobenius and separable cowreaths. Let \mathcal{C} be a monoidal category and (A, X, ψ) a cowreath in \mathcal{C} .

We say that (A, X, ψ) is Frobenius if (X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^\#$. Explicitly, according to [8, Lemma 4.4], (X, ψ) in $\mathcal{T}_A^\#$ is a Frobenius coalgebra if and only if there exist morphisms $t : \underline{1} \rightarrow AX$ and $B : X^2 \rightarrow A$ in \mathcal{C} such that

$$(1.7) \quad \begin{aligned} (a) \quad & \underline{m}_A X \circ At = \underline{m}_A X \circ A\psi \circ tA, \\ (b) \quad & \underline{m}_A \circ BA = \underline{m}_A \circ AB \circ \psi X \circ X\psi, \\ (c) \quad & \underline{m}_A X \circ A\psi \circ AXB \circ \delta X = \underline{m}_A X \circ ABX \circ \psi X^2 \circ X\delta, \\ (d) \quad & \underline{m}_A \circ AB \circ tX = \epsilon = \underline{m}_A \circ AB \circ \psi X \circ Xt. \end{aligned}$$

t is called the Frobenius element and B is called the Casimir morphism, while the pair (t, B) is called a Frobenius system for the Frobenius coalgebra (X, ψ) .

We say that (A, X, ψ) is separable if (X, ψ) is a coseparable coalgebra in $\mathcal{T}_A^\#$, i.e. a coalgebra together with a normalized Casimir morphism B . By [8, Proposition 6.4] this means that $B : X^2 \rightarrow A$ is a morphism in \mathcal{C} satisfying (1.7.b, c) and the normalized condition

$$(1.8) \quad \underline{m}_A \circ AB \circ \delta = \epsilon.$$

In Sections 2, 3 and 4 we give necessary and sufficient conditions for which some cowreaths that appear in the context of quasi-Hopf algebras are Frobenius (respectively separable). In particular, this provides examples of Frobenius (respectively separable) cowreaths (or generalized entwining structures) that are not distributive laws (or entwining structures) in the classical sense. Equivalently, this means that certain forgetful functors are Frobenius (respectively separable). Recall that Morita [21] called a functor Frobenius if it has left and right adjunctions which are naturally equivalent. For the definition of a separable functor we refer to [22], or to [23] for the case when the functor has an adjoint, which is merely our case.

In the sequel, we need the following auxiliary result. Recall that an adjunction $X \dashv Y$ in \mathcal{C} is a quadruple (X, Y, b, d) , with X, Y objects in \mathcal{C} and morphisms $b : \underline{1} \rightarrow YX$ (the counit or the coevaluation morphism) and $d : XY \rightarrow \underline{1}$ (the unit or the evaluation morphism) satisfying

$$(1.9) \quad Yd \circ bY = Y \text{ and } dX \circ Xb = X.$$

Y is called a right adjoint of X , and is denoted by $*X$. X is called a left adjoint of Y , and is denoted by Y^* .

From now on we work over a field k , and \mathcal{C} will be \mathcal{M}_k , the category of k -vector spaces. To emphasize this we will denote sometimes $\mathcal{T}_A^\#$ by $\mathcal{T}(\mathcal{M}_k)_A^\#$. If, moreover, we want to emphasize a certain monoidal category \mathcal{C} then we will write $\mathcal{T}(\mathcal{C})_A^\#$ in place of $\mathcal{T}_A^\#$.

Proposition 1.3. *If $(X, \psi) \in \mathcal{T}(\mathcal{M}_k)_A^\#$ has a left or right dual, then X is finite dimensional. Consequently, if (X, ψ) is a Frobenius coalgebra in $\mathcal{T}(\mathcal{M}_k)_A^\#$, then X is finite dimensional.*

Proof. Assume that we have an adjunction $(X, \psi) \dashv (Y, \varphi)$ in $\mathcal{T}(\mathcal{M}_k)_A^\#$, with unit $b : k \rightarrow A Y X$ and counit $d : X Y \rightarrow A$. Applying the definition of the monoidal structure of $\mathcal{T}(\mathcal{M}_k)_A^\#$, we find that the second formula in (1.9) takes the form

$$(1.10) \quad mX \circ AdX \circ \psi Y X \circ Xb = \eta X.$$

Let $b(1) = a_i \otimes y_i \otimes x_i \in A \otimes Y \otimes X$, and use the following notation for ψ :

$$(1.11) \quad \psi : X \otimes A \rightarrow A \otimes X, \quad \psi(x \otimes a) = a_\psi \otimes x^\psi,$$

where in the both cases the summation is implicitly understood. Then (1.10) can be rewritten as

$$a_{i\psi} d(x^\psi \otimes y_i) \otimes x_i = f_i(x) \otimes x_i = 1_A \otimes x,$$

for all $x \in X$, where $f_i : X \rightarrow A$, $f_i(x) = a_{i\psi}d(x^\psi \otimes y_i)$. Take a complement V of the subspace $k1_A \subset A$, so that we have $A = V \oplus k1_A$ as vector spaces. Let $p : A \rightarrow V$ and $g : A \rightarrow k1_A$ be the projections of A onto V and $k1_A$. Then $a = p(a) + \langle g, a \rangle 1_A$, for all $a \in A$, and we find that

$$(p \circ f_i)(x) \otimes x_i + 1_A \otimes \langle g \circ f_i, x \rangle x_i = 1_A \otimes x$$

in $A \otimes X = V \otimes X \oplus k1_A \otimes X$. Taking the projection of both sides onto $k1_A \otimes x$, we find that

$$1_A \otimes \langle g \circ f_i, x \rangle x_i = 1_A \otimes x,$$

and $x = \langle g \circ f_i, x \rangle x_i$, proving that X is generated by x_i 's.

The proof in the case where (X, ψ) has a left dual is similar; the second statement follows from the fact that (X, ψ) is selfdual in $\mathcal{T}_A^\#(\mathcal{M}_k)$, see [8, Remark 4.3]. \square

1.3. Quasi-bialgebras and quasi-Hopf algebras. Recall that a quasi-bialgebra over a field k is an associative unital algebra H , with a comultiplication $\Delta : H \rightarrow H \otimes H$ that is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$, called the reassociator:

$$(1.12) \quad (H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes H)(\Delta(h))\Phi^{-1},$$

for all $h \in H$. In addition, Δ is an algebra morphism and is counital via an algebra map $\varepsilon : H \rightarrow k$. Φ is a normalized 3-cocycle in the following sense:

$$(1.13) \quad (H \otimes H \otimes \Delta)(\Phi)(\Delta \otimes H \otimes H)(\Phi) = (1_H \otimes \Phi)(H \otimes \Delta \otimes H)(\Phi)(\Phi \otimes 1_H),$$

$$(1.14) \quad (H \otimes \varepsilon \otimes H)(\Phi) = 1_H \otimes 1_H.$$

The following notation is used for the reassociator Φ and its inverse Φ^{-1} :

$$\Phi = X^1 \otimes X^2 \otimes X^3 = \dots, \quad \Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = \dots \in H \otimes H \otimes H.$$

Summation is implicitly understood. We use Sweedler's notation $\Delta(h) = h_1 \otimes h_2$, for the comultiplication. Since Δ is not coassociative, we have to write

$$(\Delta \otimes H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)}, \text{ etc.}$$

A quasi-Hopf algebra is a quasi-bialgebra H together with an algebra morphism $S : H \rightarrow H^{\text{op}}$ and elements $\alpha, \beta \in H$ such that

$$(1.15) \quad S(h_1)\alpha h_2 = \varepsilon(h)\alpha \text{ and } S(h_1)\beta h_2 = \varepsilon(h)\beta$$

$$(1.16) \quad X^1\beta S(X^2)\alpha X^3 = 1 \text{ and } S(x^1)\alpha x^2\beta S(x^3) = 1.$$

The antipode of a quasi-Hopf algebra is an anti-coalgebra morphism in the following sense: there exists an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, called the Drinfeld twist or gauge transformation, such that $\varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1$ and

$$(1.17) \quad f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)),$$

for all $h \in H$. Note that f and f^{-1} can be described explicitly:

$$\begin{aligned} f &= (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \\ f^{-1} &= \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)), \end{aligned}$$

where $\gamma, \delta \in H \otimes H$ are given by the formulas

$$(1.18) \quad \gamma = S(x^1X^2)\alpha x^2X_1^3 \otimes S(X^1)\alpha x^3X_2^3 = S(X^2x_2^1)\alpha X^3x^2 \otimes S(X^1x_1^1)\alpha x^3,$$

$$(1.19) \quad \delta = X_1^1x^1\beta S(X^3) \otimes X_2^1x^2\beta S(X^2x^3) = x^1\beta S(x_2^3X^3) \otimes x^2X^1\beta S(x_1^3X^2).$$

Furthermore, $f = f^1 \otimes f^2 = F^1 \otimes F^2$ and f^{-1} have the following properties:

$$(1.20) \quad f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta \text{ and}$$

$$(1.21) \quad f^1X^1 \otimes F^1f_1^2X^2 \otimes F^2f_2^2X^3 = S(X^3)f^1F_1^1 \otimes S(X^2)f^2F_2^1 \otimes S(X^1)F^2.$$

The category \mathcal{M}_H of right H -modules over H is monoidal. The tensor product of $M, N \in \mathcal{M}_H$ is $M \otimes N$ with right H -action $(m \otimes n) \cdot h = m \cdot h_1 \otimes n \cdot h_2$, for $m \in M$, $n \in N$ and $h \in H$. The unit object is the groundfield k with H -action induced by ε . The associativity constraint is given by the formula

$$a_{M,N,P}((m \otimes n) \otimes p) = m \cdot x^1 \otimes (n \cdot x^2 \otimes p \cdot x^3),$$

for all $m \in M$, $n \in N$ and $p \in P$.

A coalgebra $(C, \Delta_C, \varepsilon_C)$ in the monoidal category \mathcal{M}_H is called a right H -module coalgebra. The comultiplication $\Delta_C : C \rightarrow C \otimes C$ is denoted by $\Delta_C(c) = c_1 \otimes c_2$. Note that, in general, C is not coassociative as a coalgebra in \mathcal{M}_k .

If H is a quasi-bialgebra with reassociator Φ then H^{op} is also a quasi-bialgebra with reassociator Φ^{-1} . If H is a quasi-Hopf algebra with bijective antipode then H^{op} is also a quasi-Hopf algebra, with antipode $S_{\text{op}} = S^{-1}$, and distinguished elements $\alpha_{\text{op}} = S^{-1}(\beta)$ and $\beta_{\text{op}} = S^{-1}(\alpha)$, where $\alpha, \beta \in H$ are as in (1.15) and (1.16).

The category of H -bimodules ${}_H\mathcal{M}_H$ is isomorphic to $\mathcal{M}_{H \otimes H^{\text{op}}}$. Since $H \otimes H^{\text{op}}$ is a quasi-bialgebra, $\mathcal{M}_{H \otimes H^{\text{op}}}$ and ${}_H\mathcal{M}_H$ are monoidal categories. A coalgebra in ${}_H\mathcal{M}_H$ is called an H -bimodule coalgebra.

1.4. (Bi)comodule algebras over quasi-Hopf algebras. Recall from [17] that a right H -comodule algebra is a unital associative algebra A together with an algebra morphism $\rho : A \rightarrow A \otimes H$, $\rho(a) = a_{(0)} \otimes a_{(1)}$, and an invertible element $\Phi_\rho \in A \otimes H \otimes H$ such that:

$$(1.22) \quad \Phi_\rho(\rho \otimes H)(\rho(a)) = (A \otimes \Delta)(\rho(a))\Phi_\rho, \text{ for all } a \in A,$$

$$(1.23) \quad (1_A \otimes \Phi)(A \otimes \Delta \otimes H)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (A \otimes H \otimes \Delta)(\Phi_\rho)(\rho \otimes H \otimes H)(\Phi_\rho),$$

$$(1.24) \quad (A \otimes \varepsilon) \circ \rho = A,$$

$$(1.25) \quad (A \otimes \varepsilon \otimes H)(\Phi_\rho) = (A \otimes H \otimes \varepsilon)(\Phi_\rho) = 1_A \otimes 1_H.$$

In a similar way we can define left comodule algebras $(B, \lambda, \Phi_\lambda)$ over H . In this situation we will denote $\lambda(b) = b_{[-1]} \otimes b_{[0]} \in H \otimes B$, for all $b \in B$, and

$$\Phi_\lambda = \tilde{X}_\lambda^1 \otimes \tilde{X}_\lambda^2 \otimes \tilde{X}_\lambda^3 = \cdots; \quad \Phi_\lambda^{-1} = \tilde{x}_\lambda^1 \otimes \tilde{x}_\lambda^2 \otimes \tilde{x}_\lambda^3 = \cdots.$$

We end this Section by recalling the notion of bicomodule algebra, as it was first introduced by Hausser and Nill in [17] under the name “quasi-commuting pair of H -coactions”.

Definition 1.4. Let H be a quasi-bialgebra. An H -bicomodule algebra A is a sextuple $(A, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ such that $(A, \lambda, \Phi_\lambda)$ is a left H -comodule algebra, (A, ρ, Φ_ρ) is a right H -comodule algebra, and $\Phi_{\lambda,\rho} \in H \otimes A \otimes H$ is an invertible element such that the following compatibility relations hold, for all $u \in A$:

$$\begin{aligned} \Phi_{\lambda,\rho}(\lambda \otimes H)(\rho(u)) &= (H \otimes \rho)(\lambda(u))\Phi_{\lambda,\rho}; \\ (1_H \otimes \Phi_{\lambda,\rho})(H \otimes \lambda \otimes H)(\Phi_{\lambda,\rho})(\Phi_\lambda \otimes 1_H) &= (H \otimes H \otimes \rho)(\Phi_\lambda)(\Delta \otimes \text{Id}_A \otimes \text{Id}_H)(\Phi_{\lambda,\rho}); \\ (1_H \otimes \Phi_\rho)(H \otimes \rho \otimes H)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H) &= (H \otimes S \otimes \Delta)(\Phi_{\lambda,\rho})(\lambda \otimes H \otimes H)(\Phi_\rho). \end{aligned}$$

It is shown in [17] that the following additional relations hold in an H -bicomodule algebra A :

$$(H \otimes A \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_A, \quad (\varepsilon \otimes A \otimes H)(\Phi_{\lambda,\rho}) = 1_A \otimes 1_H.$$

$(H, \Delta, \Delta, \Phi, \Phi, \Phi)$ is an example of an H -bicomodule algebra. If A is an H -bicomodule algebra, then $(A^{\text{op}}, \lambda, \rho, \Phi_\lambda^{-1}, \Phi_\rho^{-1}, \Phi_{\lambda,\rho}^{-1})$ is an H^{op} -bicomodule algebra.

We will use the following notation:

$$\Phi_{\lambda,\rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \bar{\Theta}^1 \otimes \bar{\Theta}^2 \otimes \bar{\Theta}^3; \quad \Phi_{\lambda,\rho}^{-1} = \theta^1 \otimes \theta^2 \otimes \theta^3 = \bar{\theta}^1 \otimes \bar{\theta}^2 \otimes \bar{\theta}^3.$$

2. FROBENIUS AND SEPARABLE PROPERTIES FOR DOI-HOPF MODULES OVER QUASI-HOPF ALGEBRAS

Doi-Hopf modules over a quasi-bialgebra have been introduced in [4]. Several types of modules, for example two-sided Hopf modules and Yetter-Drinfeld modules, are Doi-Hopf modules, see [4, 7]. The category of Doi-Hopf modules appears as the category of entwined modules over an appropriate cowreath in the category of vector spaces \mathcal{M}_k . Since the groundfield k is a \otimes -generator of \mathcal{M}_k , we can apply [8, Theorems 4.8 & 6.5], to find necessary and sufficient conditions for the Frobenius and separability property of the forgetful functor from Doi-Hopf modules to modules over the underlying algebra. These conditions can be rephrased in the special situation where the underlying algebra A is the quasi-bialgebra H .

Let A be a right H -comodule algebra and let C be a right H -module coalgebra. Let $\psi : C \otimes A \rightarrow A \otimes C$, $\psi(c \otimes a) = a_{(0)} \otimes c \cdot a_{(1)}$. Then $(C, \psi) \in \mathcal{T}_A^\#$, and (C, ψ) is a coalgebra in $\mathcal{T}_A^\#$, with comultiplication and counit given by the formulas

$$\delta : C \rightarrow A \otimes C \otimes C, \delta(c) = \tilde{X}_\rho^1 \otimes c_1 \cdot \tilde{X}_\rho^2 \otimes c_2 \cdot \tilde{X}_\rho^3; \epsilon : C \rightarrow A, \epsilon(c) = \varepsilon_C(c)1_A.$$

The category $\mathcal{M}_k(\psi)_A^C$ is just the category of Doi-Hopf modules $\mathcal{M}(H)_A^C$, as introduced in [4, 7]. A Doi-Hopf module is a right A -module with a k -linear map $\rho : M \rightarrow M \otimes C$, $\rho(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\begin{aligned} (\rho \otimes M)(\rho(m)) &= (M \otimes \Delta_C)(\rho(m)) \cdot \Phi_\rho, \\ \rho(m \cdot a) &= m_{(0)} \cdot a_{(0)} \otimes m_{(1)} \cdot a_{(1)}. \end{aligned}$$

A morphism in $\mathcal{M}(H)_A^C$ is a k -linear map that is right A -linear and right C -colinear.

2.1. Frobenius properties for Doi-Hopf modules. Let H be a quasi-bialgebra, let A be a right H -comodule algebra and let C be a right H -module coalgebra.

According to [8, Theorem 4.8], the forgetful functor $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is Frobenius if and only if $(C, \psi) \in \mathcal{T}(\mathcal{M}_k)_A^\#$ is a Frobenius coalgebra. If this is the case then by Proposition 1.3 we get C finite dimensional or, equivalently, that C has a right dual in \mathcal{M}_k . Thus F is Frobenius if and only if C is finite dimensional and one of the eight equivalent conditions (i)-(viii) in [8, Theorem 5.6] is satisfied. Condition (vii) states that there exists $t = a_i \otimes c_i \in A \otimes C$ (summation understood) such that

$$(2.1) \quad aa_i \otimes c_i = a_i a_{(0)} \otimes c_i \cdot a_{(1)},$$

for all $a \in A$, and the map

$$(2.2) \quad {}^*C \otimes A \rightarrow A \otimes C, {}^*c \otimes a \mapsto \langle {}^*c, (c_i)_2 \cdot \tilde{X}_\rho^3 \rangle a_i \tilde{X}_\rho^1 a_{(0)} \otimes (c_i)_1 \cdot \tilde{X}_\rho^2 a_{(1)}$$

is an isomorphism, where ${}^*C = \text{Hom}(C, k)$ is the right dual of C in \mathcal{M}_k .

Let H be a quasi-Hopf algebra. A simple computation tells us that

$$(2.3) \quad {}^*C \otimes A \rightarrow A \otimes {}^*C, {}^*c \otimes a \mapsto \tilde{X}_\rho^1 a_{(0)} \otimes S(\tilde{X}_\rho^2 a_{(1)}) \alpha \tilde{X}_\rho^3 \cdot {}^*c$$

is an isomorphism with inverse

$$(2.4) \quad A \otimes {}^*C \rightarrow {}^*C \otimes A, a \otimes {}^*c \mapsto a_{(1)} \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \cdot {}^*c \otimes a_{(0)} \tilde{x}_\rho^1,$$

where $\langle h \cdot {}^*c, c \rangle = \langle {}^*c, c \cdot h \rangle$, for $h \in H$, $c \in C$ and ${}^*c \in {}^*C$. We therefore obtain the following result.

Proposition 2.1. *Let H be a quasi-Hopf algebra, A a right H -comodule algebra and C a right H -module coalgebra. Then the forgetful functor $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is Frobenius if and only if C is finite dimensional and there exists $t = a_i \otimes c_i \in A \otimes C$ satisfying (2.1) and such that*

$$(2.5) \quad \kappa : A \otimes {}^*C \rightarrow A \otimes C, a \otimes {}^*c \mapsto \langle {}^*c, (c_i)_2 \cdot (\tilde{x}_\rho^2)_2 p^2 S(\tilde{x}_\rho^3) \rangle aa_i \tilde{x}_\rho^1 \otimes (c_i)_1 \cdot (\tilde{x}_\rho^2)_1 p^1$$

is an isomorphism. Here $p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3) \in H \otimes H$.

Proof. This follows from the fact that the morphism (2.5) is the composition of the morphism (2.2) and the isomorphism (2.4). We leave the verification of this detail to the reader. \square

Our next result is that Proposition 2.1 can be applied in the case where C is a Frobenius coalgebra in \mathcal{M}_H . First we need the following characterisation of Frobenius coalgebras in \mathcal{M}_H .

Proposition 2.2. *Let H be a quasi-Hopf algebra. A coalgebra C in \mathcal{M}_H is Frobenius if and only if C is finite dimensional and there exists $t \in C$ obeying*

$$(2.6) \quad t \cdot h = \varepsilon(h)t,$$

for all $h \in H$, and such that

$$(2.7) \quad \chi : {}^*C \rightarrow C, \quad \chi({}^*c) = \langle {}^*c, t_2 \cdot p^2 \rangle t_1 \cdot p^1$$

is a k -linear isomorphism. Here $p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3) \in H \otimes H$.

Proof. Step 1. A Frobenius coalgebra in \mathcal{M}_H is finite dimensional.

If C is a Frobenius coalgebra in \mathcal{M}_H , then there exists $t \in C$ and $B : C \otimes C \rightarrow k$ such that $t \cdot h = \varepsilon(h)t$, $B(c \cdot h_1 \otimes c' \cdot h_2) = \varepsilon(h)B(c \otimes c')$ and

$$B(c_2 \cdot x^2 \otimes c' \cdot x^3) c_1 \cdot x^1 = B(c \cdot X^1 \otimes c'_1 \cdot X^2) c'_2 \cdot X^3, \quad B(t \otimes c) = \varepsilon_C(c) = B(c \otimes t),$$

for all $h \in H$ and $c, c' \in C$, see [8, Definition 4.2]. In this situation, C is finite dimensional since

$$c = \varepsilon_C(c_2) c_1 = B(c_2 \otimes t) c_1 = B(c_2 \cdot x^2 \otimes t \cdot x^3) c_1 \cdot x^1 = B(c \cdot X^1 \otimes t_1 \cdot X^2) t_2 \cdot X^3,$$

for all $c \in C$.

Step 2. A finite dimensional coalgebra C in \mathcal{M}_H has a right dual in \mathcal{M}_H . ${}^*C = \text{Hom}(C, k)$, with right H -action $\langle {}^*c \cdot h, c \rangle = \langle {}^*c, c \cdot S(h) \rangle$, is a right dual of C in \mathcal{M}_H . The evaluation map d and the coevaluation map b are given by the formulas

$$d(c \otimes {}^*c) = \langle {}^*c, c \cdot \beta \rangle \text{ and } b(1) = c^j \otimes c_j \cdot \alpha,$$

where $c^j \otimes c_j \in {}^*C \otimes C$ is the finite dual basis of C as a k -vector space.

Step 3 If C is a finite dimensional coalgebra in \mathcal{M}_H , then *C is an algebra in \mathcal{M}_H and C is a right *C -module; the unit of *C is ε_C , and the multiplication is given by

$$\langle {}^*c \diamond {}^*d, c \rangle = \langle {}^*c, c_2 \cdot g^2 \rangle \langle {}^*d, c_1 \cdot g^1 \rangle,$$

where $g^1 \otimes g^2 = f^{-1} \in H \otimes H$ is the inverse of the Drinfeld twist f . The right *C -action on C is given by the formula

$$c \leftarrow {}^*c = \langle {}^*c, c_2 \cdot p^2 \rangle c_1 \cdot p^1,$$

where $p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3) \in H \otimes H$.

Applying [8, Remark 4.3], we obtain that C is Frobenius if and only if C and *C are isomorphic as right *C -modules in \mathcal{M}_H , which means that there exists a right H -linear isomorphism $\chi : {}^*C \rightarrow C$ satisfying

$$\chi({}^*c \diamond {}^*d) = \chi({}^*c) \leftarrow {}^*d,$$

for all ${}^*c, {}^*d \in {}^*C$. $t = \chi(\varepsilon_C)$ satisfies (2.6-2.7). Conversely, if t satisfies (2.6-2.7), then χ as defined in (2.7) is a right *C -linear isomorphism in \mathcal{M}_H . Further detail is left to the reader. \square

Proposition 2.3. *Let H be a quasi-Hopf algebra, A a right H -comodule algebra and C a right H -module coalgebra. If C is a Frobenius coalgebra in \mathcal{M}_H then the forgetful functor $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is Frobenius.*

Proof. Proposition 2.2 produces t satisfying (2.6-2.7). It is easy to see that $1_A \otimes t$ satisfies (2.1). It follows from (2.6) and the fact that $\varepsilon(\tilde{x}_\rho^2)\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^3 = 1_A \otimes 1_H$ that the morphism κ defined in (2.5) takes the form

$$\kappa(a \otimes {}^*c) = a \otimes \langle {}^*c, t_2 \cdot p^2 \rangle t_1 \cdot p^1 = a \otimes \chi({}^*c).$$

κ is an isomorphism since χ is an isomorphism, and we conclude from Proposition 2.1 that F is a Frobenius functor. \square

H is a right H -comodule algebra, so we can consider the category of relative Hopf modules $\mathcal{M}(H)_H^C$, also denoted as \mathcal{M}_H^C , see [10]. Theorem 2.4 generalizes [16, Theorem 3.5].

Theorem 2.4. *Let H be a quasi-Hopf algebra, let A be a right H -comodule algebra and C a right H -module coalgebra, and assume that the forgetful functor $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is Frobenius. Consider an algebra map $\zeta : A \rightarrow k$, and let $t = \zeta(a_i)c_i$, where $t = a_i \otimes c_i \in A \otimes C$ satisfies (2.1). Then*

$$(2.8) \quad \gamma : {}^*C \rightarrow C, \quad \gamma({}^*c) = \langle \zeta, \tilde{x}_\rho^1 \rangle \langle {}^*c, t_2 \cdot (\tilde{x}_\rho^2)_2 p^2 S(\tilde{x}_\rho^3) \rangle t_1 \cdot (\tilde{x}_\rho^2)_1 p^1$$

is an isomorphism. If $\tilde{\zeta} = (\zeta \otimes H) \circ \rho : A \rightarrow H$ is surjective, then C is a Frobenius coalgebra in \mathcal{M}_H . Consequently, the forgetful functor $F : \mathcal{M}(H)_H^C \rightarrow \mathcal{M}_H$ is Frobenius if and only if C is a Frobenius coalgebra in \mathcal{M}_H .

Proof. Assume that F is Frobenius. It follows from Proposition 2.1 that $\kappa : A \otimes {}^*C \rightarrow A \otimes C$, given by (2.5), is an isomorphism. κ is left A -linear, so it follows that $M \otimes {}^*C \cong M \otimes_A (A \otimes {}^*C) \cong M \otimes_A (A \otimes C) \cong M \otimes C$ are isomorphic as vector spaces, for all $M \in \mathcal{M}_A$. Since $k \in \mathcal{M}_A$ by restriction of scalars via ζ , it follows that we have an isomorphism between *C and C . This isomorphism is the map γ given in (2.8).

Assume that $\tilde{\zeta} : A \rightarrow H$ is surjective. For all $a \in A$, we have that $\zeta(a_{\langle 0 \rangle})t \cdot a_{\langle 1 \rangle} = \zeta(a)t$, and it follows that $t \cdot h = \varepsilon(h)t$, for all $h \in H$. Then we obtain that $\gamma({}^*c) = \langle {}^*c, t_2 \cdot p^2 \rangle t_1 \cdot p^1$, hence γ coincides with the map χ from Proposition 2.2, and it follows that C is a Frobenius coalgebra in \mathcal{M}_H .

Take $A = H$, and $\zeta = \varepsilon$. Then $\tilde{\zeta} = H$ is surjective, so C is a Frobenius coalgebra in \mathcal{M}_H . Conversely, if C is a Frobenius coalgebra in \mathcal{M}_H , then F is Frobenius by Proposition 2.3. \square

2.2. Separability for the category of Doi-Hopf modules. We will now study the separability of the functor F . Let C be coalgebra (in the category of vector spaces, or in the category of H -(bi)modules over a quasi-bialgebra H). The convolution product $\langle {}^*c^*d, c \rangle = ({}^*c \otimes {}^*d)\Delta_C(c)$ defines a (possibly non-associative) multiplication on *C .

The result below is a specialization of [8, Proposition 6.4 & Theorem 6.5] for the cowreath defined by our Doi-Hopf datum.

Proposition 2.5. *Let H be a quasi-bialgebra, A a right H -comodule algebra and C a right H -module coalgebra. The forgetful functor $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is separable if and only if C is a coseparable coalgebra in $\mathcal{T}(\mathcal{M}_k)_A^\#$. The coseparability of C is equivalent to conditions (i), (ii) and (iii). Under the assumption that C is finite dimensional, these conditions are also equivalent to (iv), (v), (vi) and (vii).*

(i) *There exists a k -linear map $\xi : C \otimes C \rightarrow A \otimes C$, $\xi(c \otimes c') = \xi^1(c, c') \otimes \xi^2(c, c')$, such that*

$$\begin{aligned} a_{\langle 0,0 \rangle} \xi^1(c \cdot a_{\langle 0,1 \rangle}, c' \cdot a_{\langle 1 \rangle}) \otimes \xi^2(c \cdot a_{\langle 0,1 \rangle}, c' \cdot a_{\langle 1 \rangle}) &= \xi^1(c, c') a_{\langle 0 \rangle} \otimes \xi^2(c, c') \cdot a_{\langle 1 \rangle}; \\ (\tilde{X}_\rho^1)_{\langle 0 \rangle} \xi^1(c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle}, c'_1 \cdot \tilde{X}_\rho^2) \otimes \xi^2(c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle}, c'_1 \cdot \tilde{X}_\rho^2) &\otimes c'_2 \cdot \tilde{X}_\rho^3 \\ &= \xi^1(c, c') \tilde{X}_\rho^1 \otimes \xi^2(c, c')_1 \cdot \tilde{X}_\rho^2 \otimes \xi^2(c, c')_2 \cdot \tilde{X}_\rho^3 \\ &= \tilde{X}_\rho^1 \xi^1(c_2 \cdot \tilde{X}_\rho^3, c')_{\langle 0 \rangle} \otimes c_1 \cdot \tilde{X}_\rho^2 \xi^1(c_2 \cdot \tilde{X}_\rho^3, c')_{\langle 1 \rangle} \otimes \xi^2(c_2 \cdot \tilde{X}_\rho^3, c'); \\ \tilde{X}_\rho^1 \xi^1(c_1 \cdot \tilde{X}_\rho^2, c_2 \cdot \tilde{X}_\rho^3) \otimes \xi^2(c_1 \cdot \tilde{X}_\rho^2, c_2 \cdot \tilde{X}_\rho^3) &= 1_A \otimes c, \end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(ii) There exists a k -linear map $\mathbf{B} : C \otimes C \rightarrow A$ such that

$$\begin{aligned}
 (2.9) \quad & a_{\langle 0,0 \rangle} \mathbf{B}(c \cdot a_{\langle 0,1 \rangle} \otimes c' \cdot a_{\langle 1 \rangle}) = \mathbf{B}(c \otimes c')a, \\
 & \tilde{X}_\rho^1 \mathbf{B}(c_\underline{2} \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 0 \rangle} \otimes c_\underline{1} \cdot \tilde{X}_\rho^2 \mathbf{B}(c_\underline{2} \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 1 \rangle} \\
 (2.10) \quad & = (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbf{B}(c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes c'_\underline{1} \cdot \tilde{X}_\rho^2) \otimes c'_\underline{2} \cdot \tilde{X}_\rho^3, \\
 (2.11) \quad & \tilde{X}_\rho^1 \mathbf{B}(c_\underline{1} \cdot \tilde{X}_\rho^2 \otimes c_\underline{2} \cdot \tilde{X}_\rho^3) = \varepsilon_C(c)1_A,
 \end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(iii) There exists a k -linear map $T : C \rightarrow \text{Hom}_k(C, A)$ such that

$$\begin{aligned}
 & T(c)(c')a = a_{\langle 0,0 \rangle} T(c \cdot a_{\langle 1 \rangle})(c' \cdot a_{\langle 0,1 \rangle}), \\
 & \tilde{X}_\rho^1 T(c')(c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 0 \rangle} \otimes c_\underline{1} \cdot \tilde{X}_\rho^2 T(c')(c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 1 \rangle} \\
 & = (\tilde{X}_\rho^1)_{\langle 0 \rangle} T(c'_\underline{1} \cdot \tilde{X}_\rho^2)(c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle}) \otimes c'_\underline{2} \cdot \tilde{X}_\rho^3, \\
 & \tilde{X}_\rho^1 T(c_\underline{2} \cdot \tilde{X}_\rho^3)(c_\underline{1} \cdot \tilde{X}_\rho^2) = \varepsilon_C(c)1_A,
 \end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(iv) There exists a left and right A -linear, C -colinear map $\Psi : A \otimes C \rightarrow {}^*C \otimes A$, $\Psi(a \otimes c) = \Psi^1(a \otimes c) \otimes \Psi^2(a \otimes c)$, such that

$$(2.12) \quad \Psi^1(1_A \otimes c_\underline{2} \cdot \tilde{X}_\rho^3)(c_\underline{1} \cdot \tilde{X}_\rho^2) \tilde{X}_\rho^1 \Psi^2(1_A \otimes c_\underline{2} \cdot \tilde{X}_\rho^3) = \varepsilon_C(c)1_A,$$

for all $c \in C$.

(v) There exists a left and right A -linear, C -colinear map $\Lambda : A \otimes C \rightarrow A \otimes {}^*C$, $\Lambda(c \otimes c') = \Lambda^1(c \otimes c') \otimes \Lambda^2(c \otimes c')$, such that

$$(2.13) \quad \Lambda^2(1_A \otimes c_\underline{2} \cdot \tilde{X}_\rho^3)(c_\underline{1} \cdot \tilde{X}_\rho^2 \Lambda^1(1_A \otimes c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 1 \rangle} \tilde{p}_\rho^2) \tilde{X}_\rho^1 \Lambda^1(1_A \otimes c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 0 \rangle} \tilde{p}_\rho^1 = \varepsilon_C(c)1_A,$$

for all $c \in C$. Here $\tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \tilde{x}_\rho^1 \otimes S(\tilde{x}_\rho^2) \alpha \tilde{x}_\rho^3 \in A \otimes H$. The left and right A -action and the right C -coaction on $A \otimes {}^*C$ are given by the formulas

$$(2.14) \quad a \cdot (a' \otimes {}^*c) = aa' \otimes {}^*c; \quad (a \otimes {}^*c) \cdot a' = aa'_{\langle 0 \rangle} \otimes {}^*c \cdot a'_{\langle 1 \rangle},$$

$$(2.15) \quad a \otimes {}^*c \mapsto a \tilde{X}_\rho^1 \otimes (c^j \cdot S^{-1}(\tilde{X}_\rho^3) q^2 (\tilde{X}_\rho^2)_2 S^{-1}(g^1)) ({}^*c \cdot q^1 (\tilde{X}_\rho^2)_1 S^{-1}(g^2)) \otimes c_j,$$

where $c_j \otimes c^j \in C \otimes {}^*C$ is the finite dual basis for C , and $q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2 \in H \otimes H$.

(vi) There exists a k -linear map $\bar{\Psi} : C \rightarrow {}^*C \otimes A$, $\bar{\Psi}(c) = \bar{\Psi}^1(c) \otimes \bar{\Psi}^2(c)$, such that

$$\begin{aligned}
 & \bar{\Psi}^1(c' \cdot a_{\langle 1 \rangle})(c \cdot a_{\langle 0,1 \rangle}) a_{\langle 0,0 \rangle} \bar{\Psi}^2(c' \cdot a_{\langle 1 \rangle}) = \bar{\Psi}^1(c')(c) \bar{\Psi}^2(c')a; \\
 & \bar{\Psi}^1(c')(c_\underline{2} \cdot \tilde{X}_\rho^3) \tilde{X}_\rho^1 \bar{\Psi}^2(c')_{\langle 0 \rangle} \otimes c_\underline{1} \cdot \tilde{X}_\rho^2 \bar{\Psi}^2(c')_{\langle 1 \rangle} \\
 & = \bar{\Psi}^1(c'_\underline{1} \cdot \tilde{X}_\rho^2)(c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle}) (\tilde{X}_\rho^1)_{\langle 0 \rangle} \bar{\Psi}^2(c'_\underline{1} \cdot \tilde{X}_\rho^2) \otimes c'_\underline{2} \cdot \tilde{X}_\rho^3; \\
 & \bar{\Psi}^1(c_\underline{2} \cdot \tilde{X}_\rho^3)(c_\underline{1} \cdot \tilde{X}_\rho^2) \tilde{X}_\rho^1 \bar{\Psi}^2(c_\underline{2} \cdot \tilde{X}_\rho^3) = \varepsilon_C(c)1_A,
 \end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(vii) There exists a k -linear map $\bar{\Lambda} : C \rightarrow A \otimes {}^*C$, $\bar{\Lambda}(c) = \bar{\Lambda}^1(c) \otimes \bar{\Lambda}^2(c)$, such that

$$\begin{aligned}
 & \bar{\Lambda}^1(c) a_{\langle 0 \rangle} \otimes S(a_{\langle 1 \rangle}) \cdot \bar{\Lambda}^2(c) = a_{\langle 0 \rangle} \bar{\Lambda}^1(c \cdot a_{\langle 1 \rangle}) \otimes \bar{\Lambda}^2(c \cdot a_{\langle 1 \rangle}), \\
 & \tilde{X}_\rho^1 \bar{\Lambda}^1(c_\underline{1} \cdot \tilde{X}_\rho^2) \otimes \bar{\Lambda}^2(c_\underline{1} \cdot \tilde{X}_\rho^2) \otimes c_\underline{2} \cdot \tilde{X}_\rho^3 \\
 & = \bar{\Lambda}^1(c) \tilde{X}_\rho^1 \otimes \left(g^1 S(q^2 (\tilde{X}_\rho^2)_2) \tilde{X}_\rho^3 \cdot c^j \right) \left(g^2 S(q^1 (\tilde{X}_\rho^2)_1) \cdot \bar{\Lambda}^2(c) \right) \otimes c_j, \\
 & \bar{\Lambda}^2(c_\underline{2} \cdot \tilde{X}_\rho^3)(c_\underline{1} \cdot \tilde{X}_\rho^2 \bar{\Lambda}^1(c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 1 \rangle} \tilde{p}_\rho^2) \tilde{X}_\rho^1 \bar{\Lambda}^1(c_\underline{2} \cdot \tilde{X}_\rho^3)_{\langle 0 \rangle} \tilde{p}_\rho^1 = \varepsilon_C(c)1_A,
 \end{aligned}$$

for all $c \in C$ and $a \in A$. Here $g^1 \otimes g^2$ is the inverse of the Drinfeld's twist f , $\bar{p}_\rho^1 \otimes \bar{p}_\rho^2 = \bar{x}_\rho^1 \otimes \bar{x}_\rho^2 \beta S(\bar{x}_\rho^3)$ and $c_j \otimes c^j \in C \otimes {}^*C$ is the finite dual basis for C . For simplicity, we considered *C as a left H -module via $h \cdot {}^*c = {}^*c \cdot S^{-1}(h)$, for all $h \in H$ and ${}^*c \in {}^*C$.

Proof. The equivalence between (i), (ii), (iv) and (vi) follow from [8, Proposition 6.4 & Theorem 6.5].

(ii) \Leftrightarrow (iii). Follows from the fact that $C^* \otimes A$ and $\text{Hom}_k(C, A)$ are isomorphic vector spaces if C is finite dimensional.

(iv) \Leftrightarrow (v). ${}^*C \otimes A$ and $A \otimes {}^*C$ are isomorphic as vector spaces, see (2.3-2.4). The left and right A -action and the right C -coaction on ${}^*C \otimes A$ can be transported on $A \otimes {}^*C$. A technical but straightforward computation shows that these structure maps are precisely the ones stated in (2.14) and (2.15), we leave the verification of these details to the reader. Furthermore, since Ψ can be recovered from Λ as

$$\Psi^1(a \otimes c) \otimes \Psi^2(a \otimes c) = \Lambda^2(a \otimes c) \cdot S^{-1}(\Lambda^1(a \otimes c)_{(1)} \bar{p}_\rho^2) \otimes \Lambda^1(a \otimes c)_{(0)} \bar{p}_\rho^1,$$

we find that Ψ satisfies (2.12) if and only if Λ satisfies the third condition in (vii).

The proof of (v) \Leftrightarrow (vi) is similar to the proof of (iv) \Leftrightarrow (vi). \square

Proposition 2.6 can be viewed as a separable analog of Proposition 2.3.

Proposition 2.6. *Let H be a quasi-bialgebra, let A be a right H -comodule algebra and let C be a right H -module coalgebra. If C is a coseparable coalgebra in \mathcal{M}_H then $F : \mathcal{M}(H)_A^C \rightarrow \mathcal{M}_A$ is a separable functor.*

Proof. Applying [8, Proposition 6.3] to the case where $\mathcal{C} = \mathcal{M}_H$, we obtain that C is coseparable if and only if there exists a k -linear map $B : C \otimes C \rightarrow k$ such that

$$(2.16) \quad B(c \cdot h_1 \otimes c' \cdot h_2) = \varepsilon(h) B(c \otimes c') ; \quad B(c_{\underline{1}} \otimes c_{\underline{2}}) = \varepsilon_C(c) ;$$

$$(2.17) \quad B(c \cdot X^1 \otimes c'_{\underline{1}} \cdot X^2) c'_{\underline{2}} \cdot X^3 = B(c_{\underline{2}} \cdot x^2 \otimes c' \cdot x^3) c_{\underline{1}} \cdot x^1,$$

for all $c, c' \in C$ and $h \in H$. With the help of B we construct a Casimir morphism $\mathbf{B} : C \otimes C \rightarrow A$ for the coalgebra (C, ψ) in $\mathcal{T}_A^\#$ as follows:

$$\mathbf{B}(c \otimes c') = B(c \cdot \bar{x}_\rho^2 \otimes c' \cdot \bar{x}_\rho^3) \bar{x}_\rho^1,$$

for all $c, c' \in C$. \mathbf{B} is a morphism in $\mathcal{T}_A^\#$ since

$$\begin{aligned} a_{(0,0)} \mathbf{B}(c \cdot a_{(0,1)} \otimes c' \cdot a_{(1)}) &= B(c \cdot a_{(0,1)} \bar{x}_\rho^2 \otimes c' \cdot a_{(1)} \bar{x}_\rho^3) a_{(0,0)} \bar{x}_\rho^1 \\ &\stackrel{(1.22)}{=} B(c \cdot \bar{x}_\rho^2 a_{(1)_1} \otimes c' \cdot \bar{x}_\rho^3 a_{(1)_2}) \bar{x}_\rho^1 a_{(0)} = \varepsilon(a_{(1)}) B(c \cdot \bar{x}_\rho^2 \otimes c' \cdot \bar{x}_\rho^3) \bar{x}_\rho^1 a_{(0)} \stackrel{(1.24)}{=} \mathbf{B}(c \otimes c') a, \end{aligned}$$

for all $a \in A$ and $c, c' \in C$. \mathbf{B} satisfies (2.10) since

$$\begin{aligned} &(\bar{X}_\rho^1)_{(0)} \mathbf{B}(c \cdot (\bar{X}_\rho^1)_{(1)} \otimes c'_{\underline{1}} \cdot \bar{X}_\rho^2) \otimes c'_{\underline{2}} \cdot \bar{X}_\rho^3 \\ &= B(c \cdot (\bar{X}_\rho^1)_{(1)} \bar{x}_\rho^2 \otimes c'_{\underline{1}} \cdot \bar{X}_\rho^2 \bar{x}_\rho^3) (\bar{X}_\rho^1)_{(0)} \bar{x}_\rho^1 \otimes c'_{\underline{2}} \cdot \bar{X}_\rho^3 \\ &\stackrel{(1.23)}{=} B\left(c \cdot \bar{x}_\rho^2 X^1 (\bar{X}_\rho^2)_1 \otimes c'_{\underline{1}} \cdot (\bar{x}_\rho^3)_1 X^2 (\bar{X}_\rho^2)_2\right) \bar{x}_\rho^1 \bar{X}_\rho^1 \otimes c'_{\underline{2}} \cdot (\bar{x}_\rho^3)_2 X^3 \bar{X}_\rho^3 \\ &\stackrel{(1.25)}{=} B\left(c \cdot \bar{x}_\rho^2 X^1 \otimes c'_{\underline{1}} \cdot (\bar{x}_\rho^3)_1 X^2\right) \bar{x}_\rho^1 \otimes c'_{\underline{2}} \cdot (\bar{x}_\rho^3)_2 X^3 \\ &= B(c_{\underline{2}} \cdot (\bar{x}_\rho^2)_2 x^2 \otimes c' \cdot \bar{x}_\rho^3 x^3) \bar{x}_\rho^1 \otimes c_{\underline{1}} \cdot (\bar{x}_\rho^2)_1 x^1 \\ &\stackrel{(1.23)}{=} B\left(c_{\underline{2}} \cdot \bar{Y}_\rho^3 \bar{y}_\rho^2 (\bar{x}_\rho^3)_1 \otimes c' \cdot \bar{y}_\rho^3 (\bar{x}_\rho^3)_2\right) \bar{Y}_\rho^1 (\bar{y}_\rho^1)_{(0)} \bar{x}_\rho^1 \otimes c_{\underline{1}} \cdot \bar{Y}_\rho^2 (\bar{y}_\rho^1)_{(1)} \bar{x}_\rho^2 \\ &\stackrel{(1.25)}{=} B\left(c_{\underline{2}} \cdot \bar{Y}_\rho^3 \bar{y}_\rho^2 \otimes c' \cdot \bar{y}_\rho^3\right) \bar{Y}_\rho^1 (\bar{y}_\rho^1)_{(0)} \otimes c_{\underline{1}} \cdot \bar{Y}_\rho^2 (\bar{y}_\rho^1)_{(1)} \\ &= \bar{Y}_\rho^1 \mathbf{B}(c_{\underline{2}} \cdot \bar{Y}_\rho^3 \otimes c')_{(0)} \otimes c_{\underline{1}} \cdot \bar{Y}_\rho^2 \mathbf{B}(c_{\underline{2}} \cdot \bar{Y}_\rho^3 \otimes c')_{(1)}. \end{aligned}$$

Finally, \mathbf{B} satisfies the normalizing condition (2.11), since

$$\tilde{X}_\rho^1 \mathbf{B}(c_\perp \cdot \tilde{X}_\rho^2 \otimes c_\perp \cdot \tilde{X}_\rho^3) = B(c_\perp \otimes c_\perp) 1_A = \varepsilon_C(c) 1_A,$$

for all $c \in C$. \square

We have seen in Theorem 2.4 that a right H -module coalgebra C is Frobenius in \mathcal{M}_H if the forgetful functor $F : \mathcal{M}_H^C \rightarrow \mathcal{M}_H$ is Frobenius. A similar property does not hold in general for separability. In order to conclude that C is a coseparable coalgebra from the fact that the forgetful functor is separable, an additional condition on the Casimir morphism $\mathbf{B} : C \otimes C \rightarrow H$ associated to F is needed.

Theorem 2.7. *Let H be a quasi-Hopf algebra with bijective antipode and let C be a right H -module coalgebra. There is a bijective correspondence between the set of normalized Casimir morphisms for C in \mathcal{M}_H and the set of normalized Casimir morphisms \mathbf{B} for (C, ψ) in $\mathcal{T}_H^\#$ satisfying the condition*

$$(2.18) \quad \mathbf{B}(c \otimes c') = \varepsilon \mathbf{B}(c \cdot x^2 \otimes c' \cdot x^3) x^1,$$

for all $c, c' \in C$.

Proof. A normalized Casimir morphism for C in \mathcal{M}_H is a map $B : C \otimes C \rightarrow k$ satisfying (2.16-2.17). We have seen in the proof of Proposition 2.6 that $\mathbf{B} : C \otimes C \rightarrow H$, $\mathbf{B}(c \otimes c') = B(c \cdot x^2 \otimes c' \cdot x^3) x^1$ is a normalized Casimir morphism for (C, ψ) in $\mathcal{T}_H^\#$, which means that

$$(2.19) \quad h_{(1,1)} \mathbf{B}(c \cdot h_{(1,2)} \otimes c' \cdot h_2) = \mathbf{B}(c \otimes c') h,$$

$$(2.20) \quad X^1 \mathbf{B}(c_\perp \cdot X^3 \otimes c')_1 \otimes c_\perp \cdot X^2 \mathbf{B}(c_\perp \cdot X^3 \otimes c')_2 = X_1^1 \mathbf{B}(c \cdot X_2^1 \otimes c'_\perp \cdot X^2) \otimes c'_\perp \cdot X^3,$$

$$(2.21) \quad X^1 \mathbf{B}(c_\perp \cdot X^2 \otimes c_\perp \cdot X^3) = \varepsilon_C(c) 1_H.$$

From the definition of \mathbf{B} , it follows easily that \mathbf{B} satisfies (2.18).

Conversely, assume that $\mathbf{B} : C \otimes C \rightarrow A$ satisfies (2.18-2.21). Then $B = \varepsilon \mathbf{B}$ is a normalized Casimir morphism for C in \mathcal{M}_H . It follows from (2.19) that B is a morphism in \mathcal{M}_H , and it follows from (2.21) that B is normalized, that is, $B(c_\perp \otimes c_\perp) = \varepsilon_C(c)$, for all $c \in C$. The most difficult part is to prove that B satisfies (2.17). First observe that $q_L = q^1 \otimes q^2 = S(x^1) \alpha x^2 \otimes x^3 \in H \otimes H$ has the properties

$$(2.22) \quad q_1^2 p^1 S^{-1}(q^1) \otimes q_2^2 p^2 = 1 \otimes 1 \text{ and } S(h_1) q^1 h_{(2,1)} \otimes q^2 h_{(2,2)} = q^1 \otimes h q^2,$$

for all $h \in H$. Here $p_L = p^1 \otimes p^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3 \in H \otimes H$. For later use, we record that

$$(2.23) \quad S(p^1) q_1^1 p_1^2 \otimes q_2^2 p_2^2 = 1 \otimes 1 \text{ and } h_{(2,1)} p^1 S^{-1}(h_1) \otimes h_{(2,2)} p^2 = p^1 \otimes p^2 h,$$

for all $h \in H$. We will compute

$$X = c'_\perp \cdot Y^3 y^3 S^{-1}(q^1 X^1 \mathbf{B}(c \cdot Y^1 q_1^2 X^2 \otimes c'_\perp \cdot Y^2 q_2^2 X^3) y^2 \beta) y^1$$

in two different ways.

$$\begin{aligned} X &\stackrel{(2.18)}{=} B(c \cdot Y^1 q_1^2 \otimes c'_\perp \cdot Y^2 q_2^2) c'_\perp \cdot Y^3 y^3 S^{-1}(q^1 y^2 \beta) y^1 \\ &\stackrel{(2.16)}{=} B(c \cdot Y^1 \otimes c'_\perp \cdot Y^2) c'_\perp \cdot Y^3 y^3 S^{-1}(\alpha y^2 \beta) y^1 \stackrel{(1.16)}{=} B(c \cdot Y^1 \otimes c'_\perp \cdot Y^2) c'_\perp \cdot Y^3. \end{aligned}$$

Making use of the formula $q^1 X^1 \otimes q_1^2 X^2 \otimes q_2^2 X^3 = S(z^1) q^1 z_1^2 \otimes q^2 z_2^2 \otimes z^3$, we also find that

$$\begin{aligned} X &= c'_\perp \cdot Y^3 y^3 S^{-1}(q^1 z_1^2 \mathbf{B}(c \cdot Y^1 q^2 z_2^2 \otimes c'_\perp \cdot Y^2 z^3) y^2 \beta) z^1 y^1 \\ &\stackrel{(2.19, 2.22)}{=} c'_\perp \cdot Y^3 y^3 S^{-1}(q^1 (Y_2^1 z^2 y_1^2)_1 \mathbf{B}(c \cdot q^2 (Y_2^1 z^2 y_1^2)_2 \otimes c'_\perp \cdot Y^2 z^3 y_2^2) \beta) Y_1^1 z^1 y^1 \\ &\stackrel{(1.13)}{=} ((c' \cdot x^3)_\perp \cdot X^3) \cdot S^{-1}(q^1 x_1^2 X_1^1 \mathbf{B}((c \cdot q^2 x_2^2) \cdot X_2^1 \otimes (c' \cdot x^3)_\perp \cdot X^2) \beta) x^1 \\ &\stackrel{(2.20)}{=} c_\perp \cdot q_1^2 x_{(2,1)}^2 X^2 \mathbf{B}(c_\perp \cdot q_2^2 x_{(2,2)}^2 X^3 \otimes c' \cdot x^3)_2 S^{-1}(q^1 x_1^2 X^1 \mathbf{B}(c_\perp \cdot q_2^2 x_{(2,2)}^2 X^3 \otimes c' \cdot x^3)_1 \beta) x^1 \\ &\stackrel{(1.15)}{=} B(c_\perp \cdot q_2^2 x_{(2,2)}^2 p^2 \otimes c' \cdot x^3) c_\perp \cdot q_1^2 x_{(2,1)}^2 p^1 S^{-1}(q^1 x_1^2) x^1 \end{aligned}$$

$$\begin{aligned} & \stackrel{(2.23)}{=} B(c_2 \cdot q_2^2 p^2 x^2 \otimes c' \cdot x^3) c_1 \cdot q_1^2 p^1 S^{-1}(q^1) x^1 \\ & \stackrel{(2.22)}{=} B(c_2 \cdot x^2 \otimes c' \cdot x^3) c_1 \cdot x^1. \end{aligned}$$

It follows that $B(c \cdot Y^1 \otimes c'_1 \cdot Y^2) c'_2 \cdot Y^3 = B(c_2 \cdot x^2 \otimes c' \cdot x^3) c_1 \cdot x^1$, for all $c, c' \in C$, which means that B is a normalized Casimir morphism for the coalgebra C in \mathcal{M}_H .

It can be easily checked that the two correspondences defined above are inverse each other. \square

3. FROBENIUS AND SEPARABLE PROPERTIES FOR TWO-SIDED HOPF MODULES OVER QUASI-HOPF ALGEBRAS

3.1. Two-sided Hopf modules. Consider a quasi-bialgebra H , a right H -comodule algebra A and an H -bimodule coalgebra C . A two-sided (H, A) -bimodule over C is an (H, A) -bimodule M together with a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ satisfying

$$(M \otimes \varepsilon) \rho_M = M ; \Phi \cdot (\rho_M \otimes \text{Id}_H)(\rho_M(m)) = (M \otimes \Delta)(\rho_M(m)) \cdot \Phi_\rho ;$$

$$\rho_M(h \succ m) = h_1 \succ m_{(0)} \otimes h_2 \cdot m_{(1)} ; \rho_M(m \prec a) = m_{(0)} \prec a_{(0)} \otimes m_{(1)} \cdot a_{(1)},$$

for all $m \in M$, $h \in H$ and $a \in A$. ${}_H\mathcal{M}_A^C$ is the category of two-sided (H, A) -bimodule over C , and left H -linear, right A -linear, C -colinear maps. It was shown in [6, Prop. 6.3] that ${}_H\mathcal{M}_A^C$ is isomorphic to a category of entwined modules over a cowreath in ${}_H\mathcal{M}$.

Proposition 3.1. *Let H be a quasi-bialgebra, let A be a right H -comodule algebra and let C be an H -bimodule coalgebra. $A \in {}_H\mathcal{M}$ by restriction of scalars via ε . Consider the map*

$$\psi : C \otimes A \rightarrow A \otimes C, \psi(c \otimes a) = a_{(0)} \otimes c \cdot a_{(1)}.$$

Then $(C, \psi) \in \mathcal{T}({}_H\mathcal{M})_A^\#$. (C, ψ) is a coalgebra in $\mathcal{T}({}_H\mathcal{M})_A^\#$, with comultiplication and counit

$$\delta : C \rightarrow A \otimes C \otimes C, \delta(c) = \tilde{X}_\rho^1 \otimes c_1 \cdot \tilde{X}_\rho^2 \otimes c_2 \cdot \tilde{X}_\rho^3 ; \epsilon = \eta_A \circ \varepsilon_C : C \rightarrow A.$$

The categories $({}_H\mathcal{M})(\psi)_A^C$ and ${}_H\mathcal{M}_A^C$ are isomorphic.

As before, $\Delta_C(c) = c_1 \otimes c_2$ is our Sweedler type notation for the comultiplication of a coalgebra C within the monoidal category of H -(bi)modules.

For our purposes, this description of ${}_H\mathcal{M}_A^C$ as a category of entwined modules is not very useful, since the unit object k of ${}_H\mathcal{M}$ is not \otimes -generator for ${}_H\mathcal{M}$, a condition that is needed in the more important results from [8], for example Theorem 4.8 and Proposition 6.1. This is why we provide an alternative description of ${}_H\mathcal{M}_A^C$ as a category of entwined modules, this time over a cowreath structure in ${}_k\mathcal{M}$, which will enable us to discuss when the forgetful functor $F : {}_H\mathcal{M}_A^C \rightarrow {}_H\mathcal{M}_A$ is Frobenius or separable.

Proposition 3.2. *Let H be a quasi-bialgebra, let A be a right H -comodule algebra and let C be an H -bimodule coalgebra. Then C is right $H \otimes H^{\text{op}}$ -module coalgebra and $A \otimes H^{\text{op}}$ is a right $H \otimes H^{\text{op}}$ -comodule algebra. In particular, C is a coalgebra in the category $\mathcal{T}_{A \otimes H^{\text{op}}}^\#$. Finally the category of Doi-Hopf modules $\mathcal{M}(H \otimes H^{\text{op}})_{A \otimes H^{\text{op}}}^C$ is isomorphic to the category ${}_H\mathcal{M}_A^C$ of two-sided (H, A) -bimodules over C .*

Proof. We have seen that ${}_H\mathcal{M}_H$ and $\mathcal{M}_{H \otimes H^{\text{op}}}$ are isomorphic as monoidal categories, so C is a right $H \otimes H^{\text{op}}$ -module coalgebra with right $H \otimes H^{\text{op}}$ -action $c \cdot (h \otimes h') = h' \cdot c \cdot h$, for all $c \in C$ and $h, h' \in H$. It is easy to see that $A \otimes H^{\text{op}}$ is a right $H \otimes H^{\text{op}}$ -comodule algebra with

$$\Phi_{A \otimes H^{\text{op}}} = (\tilde{X}_\rho^1 \otimes x^1) \otimes (\tilde{X}_\rho^2 \otimes x^2) \otimes (\tilde{X}_\rho^3 \otimes x^3) \in (A \otimes H^{\text{op}}) \otimes (H \otimes H^{\text{op}}) \otimes (H \otimes H^{\text{op}}),$$

and

$$\rho_{A \otimes H^{\text{op}}} : A \otimes H^{\text{op}} \rightarrow (A \otimes H^{\text{op}}) \otimes (H \otimes H^{\text{op}}), \rho_{A \otimes H^{\text{op}}}(a \otimes h) = (a_{(0)} \otimes h_1) \otimes (a_{(1)} \otimes h_2).$$

It was shown in Section 2 that C is a coalgebra in the category $\mathcal{T}_{A \otimes H^{\text{op}}}^\#$. It is now easy to verify that the categories $\mathcal{M}(H \otimes H^{\text{op}})_{A \otimes H^{\text{op}}}^C$ and ${}_H\mathcal{M}_A^C$ are isomorphic. \square

3.2. Frobenius properties for the category of two-sided Hopf modules.

Proposition 3.3. *Let H be a quasi-bialgebra, A a right H -comodule algebra and C an H -bimodule coalgebra. The forgetful functor $F : {}_H\mathcal{M}_A^C \rightarrow {}_H\mathcal{M}_A$ is Frobenius if and only if C is finite dimensional and there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ such that*

$$(3.1) \quad aa_i \otimes h_i h \otimes c_i = a_i a_{(0)} \otimes h_i h_1 \otimes h_2 \cdot c_i \cdot a_{(1)},$$

for all $a \in A$ and $h \in H$, and the k -linear map $\kappa : {}^*C \otimes A \otimes H \rightarrow A \otimes H \otimes C$,

$$(3.2) \quad \kappa({}^*c \otimes a \otimes h) = \langle {}^*c, x^3 \cdot (c_i)_2 \cdot \tilde{X}_\rho^3 \rangle a_i \tilde{X}_\rho^1 a_{(0)} \otimes h_1 x^1 h_i \otimes h_2 x^2 \cdot (c_i)_1 \cdot \tilde{X}_\rho^2 a_{(1)},$$

is an isomorphism. If H is a quasi-Hopf algebra with bijective antipode then F is Frobenius if and only if C is finite dimensional and there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ satisfying (3.1) and such that $\gamma : A \otimes H \otimes {}^*C \rightarrow A \otimes H \otimes C$ given by the formula

$$(3.3) \quad \gamma(a \otimes h \otimes {}^*c) = \langle {}^*c, (S^{-1}(X^3)q^2X_2^2 \cdot (c_i)_2 \cdot (\tilde{x}_\rho^2)_{2p^2}S(\tilde{x}_\rho^3)) \rangle aa_i \otimes h_i h \otimes q^1X_1^2 \cdot (c_i)_1 \cdot (\tilde{x}_\rho^2)_1 p^1$$

is an isomorphism. Here $q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2 \in H \otimes H$.

Proof. The conditions (3.1) and (3.2) are the conditions (2.1) and (2.2) specialized to the right $H \otimes H^{\text{op}}$ -comodule algebra $A \otimes H^{\text{op}}$. The second assertion follows from Proposition 2.1, applied to the Doi-Hopf datum in Proposition 3.2. Note that $p_R^{H^{\text{op}}} = q_R$ and $q_R^{H^{\text{op}}} = p_R$, so that $p_R^{H \otimes H^{\text{op}}} = (p^1 \otimes q^1) \otimes (p^2 \otimes q^2)$. \square

Recall that a left integral in a quasi-bialgebra H is an element $t \in H$ such that $ht = \varepsilon(h)t$, for all $h \in H$. A quasi-Hopf algebra H with bijective antipode contains a non-zero left integral if and only if H is finite dimensional, and in this case, the space of left integrals in H has dimension one, see [3]. Then there exists $\mu : H \rightarrow k$ such that $th = \langle \mu, h \rangle t$, for every $h \in H$. μ is called the modular element of H (in the dual space $H^* = {}^*H$). For all $h \in H$, we have that

$$\langle \mu, h_1 \rangle \langle \mu, S(h_2) \rangle = \langle \mu, h_1 \rangle \langle \mu, S^{-1}(h_2) \rangle = \varepsilon(h),$$

see (1.16-1.15), so it follows that μ is convolution invertible with inverse $\mu^{-1} = \mu \circ S = \mu \circ S^{-1}$. It can be easily verified that μ is an algebra map.

A right integral in H is a left integral in H^{op} . We call H unimodular if there exists a non-zero left integral in H that is also a right integral, or, equivalently, if $\mu = \varepsilon$.

Theorem 3.4. *Let H be a quasi-Hopf algebra with bijective antipode, A a right H -comodule algebra and C an H -bimodule coalgebra. Let $F : {}_H\mathcal{M}_A^C \rightarrow {}_H\mathcal{M}_A$ be the forgetful functor. Then the following assertions hold:*

- (i) *If C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, F is Frobenius.*
- (ii) *Assume that F is Frobenius. Then for every algebra morphism $\zeta : A \rightarrow k$, there exists $t \in C$ such that $h \cdot t = \varepsilon(h)t$, for all $h \in H$, $\zeta(a_{(0)})t \cdot a_{(1)} = \zeta(a)t$, for all $a \in A$, and*

$$\kappa : {}^*C \rightarrow C, \quad \kappa({}^*c) = \zeta(\tilde{x}_\rho^1) \langle {}^*c, q^2 \cdot t_2 \cdot (\tilde{x}_\rho^2)_{2p^2}S(\tilde{x}_\rho^3) \rangle q^1 \cdot t_1 \cdot (\tilde{x}_\rho^2)_1 p^1$$

is an isomorphism of vector spaces. Consequently, if there exists an algebra morphism $\zeta : A \rightarrow k$ such that $\tilde{\zeta} = (\zeta \otimes H)\rho$ is surjective then C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$.

- (iii) *$F : {}_H\mathcal{M}_H^C \rightarrow {}_H\mathcal{M}_H$ is Frobenius if and only if C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$.*
- (iv) *$F : {}_H\mathcal{M}_H^H \rightarrow {}_H\mathcal{M}_H$ is Frobenius if and only if H is finite dimensional and unimodular.*

Proof. (i) follows from Proposition 2.3 and Proposition 3.2.

(ii) If F is a Frobenius functor then there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ such that (3.1-3.3) hold. If $\zeta : A \rightarrow k$ is an algebra map then $\zeta \otimes \varepsilon : A \otimes H^{\text{op}} \rightarrow k$ is also an algebra map. Furthermore, we have that $\varepsilon(h_i h) \zeta(aa_i) c_i = \zeta(a_i a_{(0)}) h \cdot c_i \cdot a_{(1)}$, for all $h \in H$ and $a \in A$, and $t = \varepsilon(h_i) \zeta(a_i) c_i \in C$ is such that $h \cdot t = \varepsilon(h)t$, for all $h \in H$, and $\zeta(a_{(0)})t \cdot a_{(1)} = \zeta(a)t$, for all $a \in A$. The remaining assertions follow from Theorem 2.4 and Proposition 3.2.

(iii) also follows from Theorem 2.4 and Proposition 3.2.

(iv) in view of (iii), it suffices to show that $H \in {}_H\mathcal{M}_H$ is Frobenius if and only H is finite dimensional and unimodular.

If $H \in {}_H\mathcal{M}_H$ is a Frobenius coalgebra, then F is a Frobenius functor, and it follows from Proposition 3.2 and Proposition 1.3 that H is finite dimensional. H is a Frobenius coalgebra in $\mathcal{M}_{H \otimes H^{\text{op}}}$, so there exists $t \in H$ such that $ht = th = \varepsilon(h)t$, for all $h \in H$, and

$$(3.4) \quad \bar{\theta} : {}^*H \rightarrow H, \quad \bar{\theta}({}^*h) = \langle {}^*h, q^2 t_2 p^2 \rangle q^1 t_1 p^1,$$

is an isomorphism of vector spaces, see the proof of Proposition 2.3. It follows that t is a non-zero left and right integral in H , and so H is unimodular.

Conversely, assume that H is finite dimensional and unimodular. Let t be a non-zero left and right integral in H . Since t is a non-zero integral in H by [3, Remarks 2.6(ii)] we know that $\bar{\theta}$ defined in (3.4) is a left H -linear isomorphism, where *H is considered as a left H -module via $\langle h \cdot {}^*h, h' \rangle = \langle {}^*h, S^{-1}(h)h' \rangle$, for all $h, h' \in H$. But this is part of the structure of the right dual of H in ${}_H\mathcal{M}_H$, in the sense that *H is a right dual of H in ${}_H\mathcal{M}_H$. *H is an H -bimodule via $\langle h \cdot {}^*h \cdot h', k \rangle = \langle h^*, S^{-1}(h)kS(h') \rangle$, for all $h, h', k \in H$. The evaluation and coevaluation are given by the formulas

$$d : H \otimes {}^*H \rightarrow k, \quad d(h \otimes {}^*h) = \langle {}^*h, S^{-1}(\alpha)h\beta \rangle \text{ and } b : k \rightarrow {}^*H \otimes H, \quad b(1) = l^i \otimes S^{-1}(\beta)l_i\alpha,$$

where $l_i \otimes l^i \in H \otimes {}^*H$ is the finite dual basis for H .

t is also a right integral in H . Using the formula $t_1 p^1 \otimes t_2 p^2 S(h) = t_1 p^1 h \otimes t_2 p^2$, for all $h \in H$, we can prove that $\bar{\theta}$ is right H -linear. Therefore $\bar{\theta}$ is an isomorphism between H and *H in ${}_H\mathcal{M}_H$. Looking now again at the isomorphism between ${}_H\mathcal{M}_H$ and $\mathcal{M}_{H \otimes H^{\text{op}}}$ and taking into account the formulas (2.6-2.7) applied to the quasi-Hopf algebra $H \otimes H^{\text{op}}$, we conclude that H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$. \square

3.3. Separability for the category of two-sided Hopf modules. Let H be a quasi-Hopf algebra with bijective antipode, A a right H -comodule algebra and C an H -bimodule coalgebra. Then C is a coalgebra in three different monoidal categories:

- C is a coalgebra in the monoidal category ${}_H\mathcal{M}_H \cong \mathcal{M}_{H \otimes H^{\text{op}}}$, by assumption. Let \mathcal{W} be the set of normalized Casimir morphisms for C in ${}_H\mathcal{M}_H$.
- C is a coalgebra in the monoidal category $\mathcal{T}_{A \otimes H^{\text{op}}}^{\#}$, see Proposition 3.2. Let $\mathcal{W}^{\#}$ be the set of normalized Casimir morphisms for C in $\mathcal{T}_{A \otimes H^{\text{op}}}^{\#}$.
- C is a coalgebra in the monoidal category $\mathcal{T}({}_H\mathcal{M})_A^{\#}$, see Proposition 3.1. Let ${}_H\mathcal{W}^{\#}$ be the set of normalized Casimir morphisms for C in $\mathcal{T}({}_H\mathcal{M})_A^{\#}$.

Recall that, in general, a coseparable coalgebra is a coalgebra together with a normalized Casimir morphism.

It is immediate that coseparability of C as a coalgebra in ${}_H\mathcal{M}_H \cong \mathcal{M}_{H \otimes H^{\text{op}}}$ implies coseparability of C as a coalgebra in $\mathcal{T}_{A \otimes H^{\text{op}}}^{\#}$, and this yields a map $w : \mathcal{W} \rightarrow \mathcal{W}^{\#}$.

If C is coseparable as a coalgebra in $\mathcal{T}({}_H\mathcal{M})_A^{\#}$, then the forgetful functor $F : ({}_H\mathcal{M})(\psi)_A^C \cong {}_H\mathcal{M}_A^C \cong \mathcal{M}(H \otimes H^{\text{op}})_{A \otimes H^{\text{op}}}^C \rightarrow {}_H\mathcal{M}_A \cong \mathcal{M}_{A \otimes H^{\text{op}}}$ is separable, which is equivalent to the coseparability of C as a coalgebra in $\mathcal{T}_{A \otimes H^{\text{op}}}^{\#}$. This implies that we have a map $w^{\#} : {}_H\mathcal{W}^{\#} \rightarrow \mathcal{W}^{\#}$.

The aim of Theorem 3.5 is to study w and $w^{\#}$. We first give an explicit description of the elements of \mathcal{W} , $\mathcal{W}^{\#}$ and ${}_H\mathcal{W}^{\#}$.

- \mathcal{W} consists of k -linear maps $\Sigma : C \otimes C \rightarrow k$ such that

$$(3.5) \quad \Sigma(h_1 \cdot c \cdot h'_1 \otimes h_2 \cdot c' \cdot h'_2) = \varepsilon(h)\varepsilon(h')\Sigma(c \otimes c'),$$

$$(3.6) \quad \Sigma(X^2 \cdot c_2 \cdot x^2 \otimes X^3 \cdot c' \cdot x^3)X^1 \cdot c_1 \cdot x^1 = \Sigma(x^1 \cdot c \cdot X^1 \otimes x^2 \cdot c'_1 \cdot X^2)x^3 \cdot c'_2 \cdot X^3,$$

$$(3.7) \quad \Sigma(c_1 \otimes c_2) = \varepsilon_C(c),$$

for all $c, c' \in C$ and $h \in H$.

• $\mathcal{W}^\#$ consists of k -linear maps $B : C \otimes C \rightarrow A \otimes H$, $B(c \otimes c') = B^A(c \otimes c') \otimes B^H(c \otimes c') \in A \otimes H$, such that

$$(3.8) \quad \begin{aligned} & a_{\langle 0,0 \rangle} B^A(h_{(1,2)} \cdot c \cdot a_{\langle 0,1 \rangle} \otimes h_2 \cdot c' \cdot a_{\langle 1 \rangle}) \otimes B^H(h_{(1,2)} \cdot c \cdot a_{\langle 0,1 \rangle} \otimes h_2 \cdot c' \cdot a_{\langle 1 \rangle}) h_{(1,1)} \\ &= B^A(c \otimes c') a \otimes h B^H(c \otimes c'); \end{aligned}$$

$$(3.9) \quad \begin{aligned} & \tilde{X}_\rho^1 B^A(x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 0 \rangle} \otimes B^H(x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes c')_1 x^1 \otimes B^H(x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes c')_2 x^2 \cdot c_1 \\ & \cdot \tilde{X}_\rho^2 B^A(x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 1 \rangle} = (\tilde{X}_\rho^1)_{\langle 0 \rangle} B^A(x_2^1 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes x^2 \cdot c'_1 \cdot \tilde{X}_\rho^2) \\ & \otimes B^H(x_2^1 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes x^2 \cdot c'_1 \cdot \tilde{X}_\rho^2) x_1^1 \otimes x^3 \cdot c'_2 \cdot \tilde{X}_\rho^3; \end{aligned}$$

$$(3.10) \quad \tilde{X}_\rho^1 B^A(x^2 \cdot c_1 \cdot \tilde{X}_\rho^2 \otimes x^3 \cdot c_2 \cdot \tilde{X}_\rho^3) \otimes B^H(x^2 \cdot c_1 \cdot \tilde{X}_\rho^2 \otimes x^3 \cdot c_2 \cdot \tilde{X}_\rho^3) x^1 = \varepsilon_C(c) 1_A \otimes 1_H,$$

for all $c, c' \in C$, $a \in A$ and $h \in H$.

• ${}_H \mathcal{W}^\#$ consists of k -linear maps $\mathbb{B} : C \otimes C \rightarrow A$ such that

$$(3.11) \quad a_{\langle 0,0 \rangle} \mathbb{B}(h_1 \cdot c \cdot a_{\langle 0,1 \rangle} \otimes h_2 \cdot c' \cdot a_{\langle 1 \rangle}) = \varepsilon(h) \mathbb{B}(c \otimes c') a,$$

$$(3.12) \quad \begin{aligned} & (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbb{B}(x^1 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes x^2 \cdot c'_1 \cdot \tilde{X}_\rho^2) \otimes x^3 \cdot c'_2 \cdot \tilde{X}_\rho^3 \\ &= \tilde{X}_\rho^1 \mathbb{B}(X^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 0 \rangle} \otimes X^1 \cdot c_1 \cdot \mathbb{B}(X^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 1 \rangle} \end{aligned}$$

$$(3.13) \quad \tilde{X}_\rho^1 \mathbb{B}(c_1 \cdot \tilde{X}_\rho^2 \otimes c_2 \cdot \tilde{X}_\rho^3) = \varepsilon_C(c) 1_A,$$

for all $c, c' \in C$, $a \in A$ and $h \in H$.

Theorem 3.5. *Let H be a quasi-Hopf algebra with bijective antipode, A a right H -comodule algebra and C an H -bimodule coalgebra.*

(i) *We have a map $w : \mathcal{W} \rightarrow \mathcal{W}^\#$,*

$$w(\Sigma)(c \otimes c') = \Sigma(X^2 \cdot c \cdot \tilde{x}_\rho^2 \otimes X^3 \cdot c' \cdot \tilde{x}_\rho^3) \tilde{x}_\rho^1 \otimes X^1.$$

In the case where $A = H$, w corestricts to a bijection between \mathcal{W} and the subset $\underline{\mathcal{W}}^\# \subset \mathcal{W}^\#$ consisting of $B \in \mathcal{W}^\#$ satisfying

$$B(c \otimes c') = (\varepsilon \otimes \varepsilon) B(X^2 \cdot c \cdot x^2 \otimes X^3 \cdot c' \cdot x^3) x^1 \otimes X^1,$$

for all $c, c' \in C$. The inverse of w is $w^{-1}(B) = (\varepsilon \otimes \varepsilon) B$.

(ii) *We have a map $w^\# : {}_H \mathcal{W}^\# \rightarrow \mathcal{W}^\#$,*

$$w^\#(\mathbb{B})(c \otimes c') = \mathbb{B}(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1.$$

$w^\#$ corestricts to an isomorphism between ${}_H \mathcal{W}^\#$ and the subset $\underline{\underline{\mathcal{W}}}^\# \subset \mathcal{W}^\#$ consisting of $B \in \mathcal{W}^\#$ satisfying

$$B(c \otimes c') = (A \otimes \varepsilon) B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1,$$

for all $c, c' \in C$. The inverse of $w^\#$ is $(w^\#)^{-1}(B) = (A \otimes \varepsilon) \circ B$.

(iii) *If $A = H$, then $\underline{\mathcal{W}}^\# \subset \underline{\underline{\mathcal{W}}}^\#$. Consequently, if C is a coseparable coalgebra in ${}_H \mathcal{M}_H$, then it is a coseparable coalgebra in $\mathcal{T}({}_H \mathcal{M})_H^\#$, too.*

Proof. (i) Follows easily from Theorem 2.7 applied to the data in Proposition 3.2.

(ii) We first show that $B = w(\Sigma) \in \mathcal{W}^\#$. B satisfies (3.9) since

$$\begin{aligned} & \tilde{X}_\rho^1 \mathbb{B}(X^2 x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 0 \rangle} \otimes X_1^1 x^1 \otimes X_2^1 x^2 \cdot c_1 \cdot \tilde{X}_\rho^2 \mathbb{B}(X^2 x^3 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 1 \rangle} \\ & \stackrel{(1.13)}{=} \tilde{X}_\rho^1 \mathbb{B}(x_1^3 X^2 Y_2^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes x_2^3 X^3 Y^3 \cdot c')_{\langle 0 \rangle} \otimes x^1 Y^1 \\ & \quad \otimes x^2 X^1 Y_1^2 \cdot c_1 \cdot \tilde{X}_\rho^2 \mathbb{B}(x_1^3 X^2 Y_2^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes x_2^3 X^3 Y^3 \cdot c')_{\langle 1 \rangle} \\ & \stackrel{(3.11)}{=} \tilde{X}_\rho^1 \mathbb{B}(X^2 \cdot (Y^2 \cdot c)_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot (Y^3 \cdot c'))_{\langle 0 \rangle} \otimes Y^1 \end{aligned}$$

$$\begin{aligned}
 & \otimes X^1 \cdot (Y^2 \cdot c)_{\underline{1}} \cdot \tilde{X}_\rho^2 \mathbb{B}(X^2 \cdot (Y^2 \cdot c)_{\underline{2}} \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot (Y^3 \cdot c'))_{\langle 1 \rangle} \\
 \stackrel{(3.12)}{=} & (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbb{B}(x^1 Y^2 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes x^2 Y_1^3 \cdot c'_1 \cdot \tilde{X}_\rho^2) \otimes Y^1 \otimes x^3 Y_2^3 \cdot c'_2 \cdot \tilde{X}_\rho^3 \\
 \stackrel{(3.11)}{=} & (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbb{B}(y_1^2 x^1 Y^2 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes y_2^2 x^2 Y_1^3 \cdot c'_1 \cdot \tilde{X}_\rho^2) \otimes y^1 Y^1 \otimes y^3 x^3 Y_2^3 \cdot c'_2 \cdot \tilde{X}_\rho^3 \\
 \stackrel{(1.13)}{=} & (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbb{B}(X^2 x_2^1 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes X^3 x^2 \cdot c'_1 \cdot \tilde{X}_\rho^2) \otimes X^1 x_1^1 \otimes x^3 \cdot c'_2 \cdot \tilde{X}_\rho^3.
 \end{aligned}$$

It is left to the reader to show that (3.11, 1.12) imply (3.8), and that (3.13) implies (3.10). Moreover, $B(c \otimes c') = (A \otimes \varepsilon)B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1$, so that $\text{Im}(w^\#) \subset \underline{\mathcal{W}}^\#$.

Conversely, if $B \in \underline{\mathcal{W}}^\#$, then $\mathbb{B} = (A \otimes \varepsilon) \circ B \in {}_H\mathcal{W}^\#$. Applying $A \otimes \varepsilon \otimes C$ to (3.9) we find that

$$\begin{aligned}
 & (\tilde{X}_\rho^1)_{\langle 0 \rangle} \mathbb{B}(x^1 \cdot c \cdot (\tilde{X}_\rho^1)_{\langle 1 \rangle} \otimes x^2 \cdot c'_1 \cdot \tilde{X}_\rho^2) \otimes x^3 \cdot c'_2 \cdot \tilde{X}_\rho^3 \\
 &= \tilde{X}_\rho^1 B^A(c_2 \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 0 \rangle} \otimes B^H(c_2 \cdot \tilde{X}_\rho^3 \otimes c') \cdot c_1 \cdot \tilde{X}_\rho^2 B^A(c_2 \cdot \tilde{X}_\rho^3 \otimes c')_{\langle 1 \rangle} \\
 &\stackrel{(*)}{=} \tilde{X}_\rho^1 \mathbb{B}(X^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 0 \rangle} \otimes X^1 \cdot c_1 \cdot \tilde{X}_\rho^2 \mathbb{B}(X^2 \cdot c_2 \cdot \tilde{X}_\rho^3 \otimes X^3 \cdot c')_{\langle 1 \rangle},
 \end{aligned}$$

as needed. At (*), we used the formula

$$B(c \otimes c') = (A \otimes \varepsilon)B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1 = \mathbb{B}(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1,$$

for all $c, c' \in C$. Applying $A \otimes \varepsilon$ to (3.8) and (3.10), we obtain (3.11) and (3.13), proving that $\mathbb{B} \in {}_H\mathcal{W}^\#$. It is straightforward to see that both constructions are inverses.

(iii) If $B \in \underline{\mathcal{W}}^\#$ then $B \in \underline{\mathcal{W}}^\#$ since

$$\begin{aligned}
 & (H \otimes \varepsilon)B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1 \\
 &= (\varepsilon \otimes \varepsilon)B(Y^2 X^2 \cdot c \cdot x^2 \otimes Y^3 X^3 \cdot c' \cdot x^3)(H \otimes \varepsilon)(x^1 \otimes Y^1) \otimes X^1 \\
 &= (\varepsilon \otimes \varepsilon)B(X^2 \cdot c \cdot x^2 \otimes X^3 \cdot c' \cdot x^3)x^1 \otimes X^1 = B(c \otimes c'),
 \end{aligned}$$

for all $c, c' \in C$. □

We will now focus on the situation where $A = C = H$. The coseparability of H as a coalgebra in ${}_H\mathcal{M}_H$ has been studied in [18, Sec. 7]. Following [18], a biinvariant form is a morphism $\Xi : H \otimes H \rightarrow k$ in ${}_H\mathcal{M}_H$. Ξ is called cocentral if

$$\Xi(X^2 h_2 x^2 \otimes X^3 h'_1 x^3) X^1 h_1 x^1 = \Xi(x^1 h X^1 \otimes x^2 h'_1 X^2) x^3 h'_2 X^3,$$

for all $h, h' \in H$. Thus a biinvariant cocentral form in the sense of [18] is a Casimir morphism for H as a coalgebra in ${}_H\mathcal{M}_H$.

The comments following [18, Cor. 7.5] entail that the existence of a non-zero biinvariant cocentral form $\Xi : H \otimes H \rightarrow k$ for a quasi-Hopf algebra H with bijective antipode is equivalent to the unimodularity of H . In this situation, there is a bijective correspondence between biinvariant cocentral forms and left cointegrals λ on H , these are functionals $\lambda : H \rightarrow k$ satisfying

$$(3.14) \quad \lambda(V^2 h_2 U^2) V^1 h_1 U^1 = \mu(x^1) \lambda(h S(x^2)) x^3,$$

for all $h \in H$, where μ is the modular element of H and

$$(3.15) \quad U = U^1 \otimes U^2 = g^1 S(q^2) \otimes g^2 S(q^1) \text{ and } V = V^1 \otimes V^2 = S^{-1}(f^2 p^2) \otimes S^{-1}(f^1 p^1).$$

$f = f^1 \otimes f^2$ is the Drinfeld twist with inverse $f^{-1} = g^1 \otimes g^2$, $q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2$ and $p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3)$.

If H is finite dimensional then the space \mathcal{L} of left cointegrals is of dimension one. H is cosemisimple if and only if there exists $\lambda \in \mathcal{L}$ such that $\lambda(S^{-1}(\alpha)\beta) = 1$, see [18]. This leads to the following result.

Proposition 3.6. *Let H be a finite dimensional quasi-Hopf algebra. H is coseparable as a coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple.*

Proof. We have seen above that there exists a Casimir morphism for the coalgebra H in ${}_H\mathcal{M}_H$ if and only if H is unimodular. In this case, we have a bijection between \mathcal{L} and the set of Casimir morphisms for H as a coalgebra in ${}_H\mathcal{M}_H$, see [18, Cor. 7.4]. The Casimir morphism Ξ corresponding to $\lambda \in \mathcal{L}$ is given by the formula

$$\Xi(h \otimes h') = \lambda(S^{-1}(\alpha)h\beta S(h')).$$

Thus H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and there exists $\lambda \in \mathcal{L}$ such that $\lambda(S^{-1}(\alpha)h_1\beta S(h_2)) = \varepsilon(h)$, for all $h \in H$, or, equivalently, $\lambda(S^{-1}(\alpha)\beta) = 1$, that is, H is cosemisimple. \square

We will now study the separability of the forgetful functor $F : {}_H\mathcal{M}_H^H \rightarrow {}_H\mathcal{M}_H$ in the case where H is a finite dimensional quasi-Hopf algebra.

Theorem 3.7. *For a finite dimensional quasi-Hopf algebra H , the following assertions are equivalent:*

- (i) *the forgetful functor $F : {}_H\mathcal{M}_H^H \rightarrow {}_H\mathcal{M}_H$ is separable;*
- (ii) *H is unimodular;*
- (iii) *F is a Frobenius functor.*

Proof. $(i) \Rightarrow (ii)$. Suppose that F is separable. Applying Proposition 2.5 (v) and Proposition 3.2 we obtain a morphism $\Lambda : H \otimes H^{\text{op}} \otimes H \rightarrow H \otimes H^{\text{op}} \otimes {}^*H$ in $\mathcal{M}(H \otimes H^{\text{op}})_{H \otimes H^{\text{op}}}^H$ which is left $H \otimes H^{\text{op}}$ -linear and satisfies a certain normalizing condition. Then Λ is left $H \otimes H^{\text{op}}$ -linear and a morphism in ${}_H\mathcal{M}_H^H$. As in the proof of Theorem 2.4 we deduce that for any right $H \otimes H^{\text{op}}$ -module M (or, equivalently, for any H -bimodule M), there exists a morphism $M \otimes H \rightarrow M \otimes {}^*H$ in ${}_H\mathcal{M}_H^H$. Applying Proposition 2.5 (v) to the Doi-Hopf datum described in Proposition 3.2, we find that $M \otimes H$, $M \otimes {}^*H \in {}_H\mathcal{M}_H^H$. The structure maps on $M \otimes H$ and $M \otimes {}^*H$ are given by the formulas

$$\begin{aligned} h \cdot (m \otimes \bar{h}) \cdot h' &= h_1 \cdot m \cdot h'_2 \otimes h_2 \bar{h} h'_2; \\ \rho(m \otimes \bar{h}) &= x^1 \cdot m \cdot X^1 \otimes x^2 h_1 X^2 \otimes x^3 h_2 X^3; \\ h \cdot (m \otimes {}^*h) \cdot h' &= h_1 \cdot m \cdot h'_1 \otimes S(h'_2) \rightharpoonup^* h \lleftarrow S^{-1}(h_2); \\ \rho(m \otimes {}^*h) &= x^1 \cdot m \cdot X^1 \otimes (S(X^2)_1 U^1 X^3 \rightharpoonup h^i \lleftarrow x^3 V^1 S^{-1}(x^2)_1) \\ &\quad (S(X^2)_2 U^2 \rightharpoonup^* h \lleftarrow V^2 S^{-1}(x^2)_2) \otimes h_i, \end{aligned}$$

for all $h, \bar{h}, h' \in H$, ${}^*h \in {}^*H$ and $m \in M$. Here $h_i \otimes h^i \in H \otimes {}^*H$ is the finite dual basis of H , U and V are the elements defined in (3.15) and \rightharpoonup and \lleftarrow are the well-known canonical left and right actions of H on *H , given by the formula $\langle h \rightharpoonup^* h \lleftarrow h', h'' \rangle = \langle {}^*h, h'' h' h \rangle$.

Applying this to $M = k \in {}_H\mathcal{M}$ by restriction of scalars via ε , we obtain a morphism $H \rightarrow {}^*H$ in ${}_H\mathcal{M}_H^H$. $H \in {}_H\mathcal{M}_H^H$ via the multiplication and comultiplication, and the structure on *H is given by the formulas

$$(3.16) \quad h \cdot {}^*h \cdot h' = S(h') \rightharpoonup^* h \lleftarrow S^{-1}(h) \text{ and } \rho({}^*h) = (U^1 \rightharpoonup h^i \lleftarrow V^1) (U^2 \rightharpoonup^* h \lleftarrow V^2) \otimes h_i,$$

for all $h, h' \in H$ and ${}^*h \in {}^*H$. Applying [18, Theorem 7.3], we find a Casimir morphism for H as a coalgebra in ${}_H\mathcal{M}_H$, and therefore H is unimodular.

$(ii) \Rightarrow (i)$. If H is unimodular, then there exists a non-zero left and right integral t in H and a non-zero left cointegral λ on H such that $\langle \lambda, S^{-1}(t) \rangle = 1$. By [5, Prop. 4.1] we have that

$$(3.17) \quad \lambda(q^2 t_2 p^2) q^1 t_1 p^1 = \lambda(S^{-1}(q^1 t_1 p^1)) q^2 t_2 p^2 = 1.$$

In the sequel we will need the following formula, see [3, Lemma 3.3]:

$$(3.18) \quad \lambda(S^{-1}(h)h') = \mu(h_1)\lambda(h'S(h_2)),$$

for all $h, h' \in H$. So when H is unimodular we have $\lambda(S^{-1}(h)h') = \lambda(h'S(h))$, or, equivalently,

$$(3.19) \quad \lambda \leftarrow S^{-1}(h) = S(h) \rightarrow \lambda,$$

for all $h \in H$.

It follows from Proposition 2.5 (vii) and Proposition 3.2 that the separability of F is equivalent to the existence of a k -linear map $\bar{\Lambda} : H \rightarrow H \otimes H \otimes {}^*H$, $\bar{\Lambda}(h) = \bar{\Lambda}^1(h) \otimes \bar{\Lambda}^2(h) \otimes \bar{\Lambda}^3(h)$ such that:

$$(3.20) \quad \begin{aligned} & \bar{\Lambda}^1(h)h_1 \otimes h'_1\bar{\Lambda}^2(h) \otimes S(h_2) \rightarrow \bar{\Lambda}^3(h) \leftarrow S^{-1}(h'_2) \\ &= h_1\bar{\Lambda}^1(h'_2hh_2) \otimes \bar{\Lambda}^2(h'_2hh_2)h'_1 \otimes \bar{\Lambda}^3(h'_2hh_2), \end{aligned}$$

$$(3.21) \quad \begin{aligned} & \bar{\Lambda}^3(x^2h_1X^2)(S^{-1}(Y^2)h'S(y^2))X^1\bar{\Lambda}^1(x^2h_1X^2)y^1 \otimes Y^1\bar{\Lambda}^2(x^2h_1X^2)x^1 \otimes Y^3x^3h_2X^3y^3 \\ &= \bar{\Lambda}^3(h)(V^2h'_2U^2)\bar{\Lambda}^1(h) \otimes \bar{\Lambda}^2(h) \otimes V^1h'_1U^1, \end{aligned}$$

$$(3.22) \quad \begin{aligned} & \bar{\Lambda}^3(x^3h_2X^3)(q^2\bar{\Lambda}^2(x^3h_2X^3)_2x^2h_1X^2\bar{\Lambda}^1(x^3h_2X^3)_2p^2)X^1\bar{\Lambda}^1(x^3h_2X^3)_1p^1 \\ & \otimes q^1\bar{\Lambda}^2(x^3h_2X^3)_1x^1 = \varepsilon(h)1 \otimes 1, \end{aligned}$$

for all $h, h, h' \in H$. We will show that

$$\bar{\Lambda} : H \rightarrow H \otimes H \otimes {}^*H, \quad \bar{\Lambda}(h) = 1 \otimes t \otimes S(h) \rightarrow \lambda$$

satisfies (3.20-3.22). (3.20) is satisfied since

$$\begin{aligned} & \bar{\Lambda}^1(h)h_1 \otimes h'_1\bar{\Lambda}^2(h) \otimes S(h_2) \rightarrow \bar{\Lambda}^3(h) \leftarrow S^{-1}(h'_2) \\ &= h_1 \otimes h'_1t \otimes S(hh_2) \rightarrow \lambda \leftarrow S^{-1}(h'_2) \stackrel{(3.19)}{=} h_1 \otimes t \otimes S(h'h_2) \rightarrow \lambda \\ &= h_1 \otimes th'_1 \otimes S(h'_2hh_2) \rightarrow \lambda = h_1\bar{\Lambda}^1(h'_2hh_2) \otimes \bar{\Lambda}^2(h'_2hh_2)h'_1 \otimes \bar{\Lambda}^3(h'_2hh_2), \end{aligned}$$

for all $h, h, h' \in H$. To prove (3.21), we first compute that

$$\begin{aligned} & \bar{\Lambda}^3(x^2h_1X^2)(S^{-1}(Y^2)h'S(y^2))X^1\bar{\Lambda}^1(x^2h_1X^2)y^1 \otimes Y^1\bar{\Lambda}^2(x^2h_1X^2)x^1 \otimes Y^3x^3h_2X^3y^3 \\ &= (S(x^2h_1X^2) \rightarrow \lambda)(S^{-1}(Y^2)h'S(y^2))X^1y^1 \otimes Y^1tx^1 \otimes Y^3x^3h_2X^3y^3 \\ &= \lambda(h'S(h_1X^2y^2))X^1y^1 \otimes t \otimes h_2X^3y^3 = \lambda(h'S(h_1))1 \otimes t \otimes h_2, \end{aligned}$$

for all $h, h' \in H$. From [18, Lemma 3.13], we recall that the formula

$$(3.23) \quad U[1 \otimes S(h)] = \Delta(S(h_1))U[h_2 \otimes 1],$$

holds for all $h \in H$. This allows us to compute that

$$\begin{aligned} & \bar{\Lambda}^3(h)(V^2h'_2U^2)\bar{\Lambda}^1(h) \otimes \bar{\Lambda}^2(h) \otimes V^1h'_1U^1 = (S(h) \rightarrow \lambda)(V^2h'_2U^2)1 \otimes t \otimes V^1h'_1U^1 \\ &= \lambda(V^2(h'S(h_1))_2U^2)1 \otimes t \otimes V^1(h'S(h_1))_1U^1h_2 \stackrel{(3.14)}{=} \lambda(h'S(h_1))1 \otimes t \otimes h_2, \end{aligned}$$

for all $h, h' \in H$, and (3.21) follows. Finally

$$\begin{aligned} & \bar{\Lambda}^3(x^3h_2X^3)(q^2\bar{\Lambda}^2(x^3h_2X^3)_2x^2h_1X^2\bar{\Lambda}^1(x^3h_2X^3)_2p^2)X^1\bar{\Lambda}^1(x^3h_2X^3)_1p^1 \otimes q^1\bar{\Lambda}^2(x^3h_2X^3)_1x^1 \\ &= (S(x^3h_2X^3) \rightarrow \lambda)(q^2t_2x^2h_1X^2p^2)X^1p^1 \otimes q^1t_1x^1 \\ &= \lambda(q^2t_2x^2h_1X^2p^2S(x^3h_2X^3))X^1p^1 \otimes q^1t_1x^1 = \lambda(q^2t_2x^2h_1\beta S(x^3h_2))1 \otimes q^1t_1x^1 \\ &= \varepsilon(h)\lambda(q^2t_2p^2)1 \otimes q^1t_1p^1 \stackrel{(3.17)}{=} \varepsilon(h)1 \otimes 1, \end{aligned}$$

for all $h \in H$, so that (3.22) holds. □

Our conclusion is that the forgetful functor $F : {}_H\mathcal{M}_H^H \rightarrow {}_H\mathcal{M}_H$ is Frobenius if and only if H is a Frobenius coalgebra within ${}_H\mathcal{M}_H$, that is, if and only if H is finite dimensional and unimodular. But in the separable case we have a completely different situation, provided that H is a finite dimensional quasi-Hopf algebra: H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple while F is separable if and only if H is unimodular; so we might have F separable

although H is not a coseparable coalgebra in ${}_H\mathcal{M}_H$. However the remarkable thing is that in the finite dimensional case F is separable if and only if it is Frobenius, since both properties reduce at the unimodularity property of H .

4. FROBENIUS AND SEPARABLE PROPERTIES FOR YETTER-DRINFELD MODULES OVER QUASI-HOPF ALGEBRAS

The aim of this Section is to apply our results to the category of Yetter-Drinfeld modules over a quasi-Hopf algebra. In particular, we will be able to characterize when the algebra extension $H \rightarrow D(H)$, from a finite dimensional quasi-Hopf algebra H to its Drinfeld double $D(H)$ is Frobenius or separable.

Following [9] we introduce the notion of right Yetter-Drinfeld module over a quasi-bialgebra.

Definition 4.1. Let H be a quasi-bialgebra, let C be an H -bimodule coalgebra and let A an H -bicomodule algebra. A right (H, A, C) -Yetter-Drinfeld module is a right A -module M together with a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, called the right C -coaction on M , such that $\varepsilon(m_{(1)})m_{(0)} = m$ and

$$\begin{aligned} (m_{(0)} \cdot \Theta^2)_{(0)} \otimes \Theta^1 \cdot (m_{(0)} \cdot \Theta^2)_{(1)} \otimes m_{(1)} \cdot \Theta^3 \\ = (m \cdot \tilde{X}_\lambda^3)_{(0)} \cdot \tilde{X}_\rho^1 \otimes \tilde{X}_\lambda^1 \cdot (m \cdot \tilde{X}_\lambda^3)_{(1)_1} \cdot \tilde{X}_\rho^2 \otimes \tilde{X}_\lambda^2 \cdot (m \cdot \tilde{X}_\lambda^3)_{(1)_2} \cdot \tilde{X}_\rho^3; \\ m_{(0)} \cdot u_{<0>} \otimes m_{(1)} \cdot u_{<1>} = (m \cdot u_{[0]})_{(0)} \otimes u_{[-1]} \cdot (m \cdot u_{[0]})_{(1)}, \end{aligned}$$

for all $m \in M$ and $u \in A$. $\mathcal{YD}(H)_A^C$ will be the category of right (H, A, C) -Yetter-Drinfeld modules and A -linear maps preserving the C -coaction.

The category of left-right Yetter-Drinfeld modules is isomorphic to a certain category of left-right Doi-Hopf modules, see [7]. A similar result for the category of right Yetter-Drinfeld modules can easily be deduced from this. For a right H -comodule algebra A , and a left H -module coalgebra C , we have an isomorphism of categories ${}_A\mathcal{M}(H)^C \cong \mathcal{M}(H^{\text{op}})_{A^{\text{op}}}^C$. For a right Yetter-Drinfeld datum (H, A, C) as in Definition 4.1, we have an isomorphism of categories

$$(4.1) \quad \mathcal{YD}(H)_A^C \cong {}_{A^{\text{op}}}\mathcal{YD}(H)^C.$$

Combining these properties with [7, Theorem 3.8], we obtain the following isomorphisms of categories:

$$\mathcal{YD}(H)_A^C \cong {}_{A^{\text{op}}}\mathcal{YD}(H^{\text{op}})^C \cong {}_{A^{\text{op}2}}\mathcal{M}(H \otimes H^{\text{op}})^C \cong \mathcal{M}(H^{\text{op}} \otimes H)_{A^{\text{op}2\text{op}}}^C,$$

where $A^{\text{op}2}$ is the right $H \otimes H^{\text{op}}$ -comodule algebra associated to the H^{op} -bicomodule algebra A^{op} as in [7, Prop. 3.3]; the H -bimodule coalgebra C is viewed as a right $H^{\text{op}} \otimes H$ -module coalgebra through the monoidal isomorphism of categories identification ${}_H\mathcal{M}_H \cong \mathcal{M}_{H^{\text{op}} \otimes H}$. More precisely, if we denote $\underline{A}^2 = A^{\text{op}2\text{op}}$ then $\underline{A}^2 = A$ as a k -algebra and it is a right $H^{\text{op}} \otimes H$ -comodule algebra with coaction

$$\underline{\rho}^2 : A \rightarrow A \otimes (H^{\text{op}} \otimes H), \quad \underline{\rho}^2(u) = u_{[0]_{(0)}} \otimes (S(u_{[-1]}) \otimes u_{[0]_{(1)}});$$

and

$$\Phi_{\underline{\rho}^2} = (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \Theta_{(0)}^2 \otimes (S(\tilde{x}_\lambda^2 \Theta)^1 f^1 \otimes (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \Theta_{(1)}^2) \otimes (S(\tilde{x}_\lambda^1) f^2 \otimes (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3).$$

C is a right $H^{\text{op}} \otimes H$ -module coalgebra, with right $H^{\text{op}} \otimes H$ -action given by $c \cdot (h \otimes h') = h \cdot c \cdot h'$, for all $h, h' \in H$ and $c \in C$.

By the opposite versions of [7, Lemmas 3.6 and 3.7], we have that the category isomorphism $F : \mathcal{YD}(H)_A^C \rightarrow \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{A}^2}^C$ is given by the following formulas. $F(M) = M$ as a right A -module, and the right C -coaction is given by

$$\rho'_M(m) = m_{(0')} \otimes m_{(1')} = (m \cdot \tilde{q}_\lambda^2)_{(0)} \otimes \tilde{q}_\lambda^1 \cdot (m \cdot \tilde{q}_\lambda^2)_{(1)},$$

for all $m \in M$. Here $\tilde{q}_\lambda = \tilde{q}_\lambda^1 \otimes \tilde{q}_\lambda^2 = S(\tilde{x}_\lambda^1) \alpha \tilde{x}_\lambda^2 \otimes \tilde{x}_\lambda^3 \in H \otimes A$.

We present a description of the inverse G of F . For a right $(H^{\text{op}} \otimes H, \underline{A}^2, C)$ -Hopf module M , with right A -action \cdot and right C -coaction ρ'_M , $\rho'_M(m) = m_{(0')} \otimes m_{(1')} \in M \otimes C$, we define $G(M) = M$ as a right A -module, with right C -coaction $\bar{\rho}_M : M \rightarrow M \otimes C$, given by the formula

$$(4.2) \quad \bar{\rho}_M(m) = m_{\overline{(0)}} \otimes m_{\overline{(1)}} = m_{(0')} \cdot (\tilde{p}_\lambda^2)_{\langle 0 \rangle} \otimes S(\tilde{p}_\lambda^1) \cdot m_{(1')} \cdot (\tilde{p}_\lambda^2)_{\langle 1 \rangle},$$

for all $m \in M$. Here $\tilde{p}_\lambda = \tilde{p}_\lambda^1 \otimes \tilde{p}_\lambda^2 = \tilde{X}_\lambda^2 S^{-1}(\tilde{X}_\lambda^1 \beta) \otimes \tilde{X}_\lambda^3 \in H \otimes A$. G is the identity on morphisms. The study of the Frobenius and separable properties of the forgetful functor $F : \mathcal{YD}(H)_A^C \rightarrow \mathcal{M}_A$ reduces to the study of the Frobenius and separable properties of the forgetful functor

$$F : \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{A}^2}^C \rightarrow \mathcal{M}_A.$$

Thus necessary and sufficient conditions for the Frobenius property or the separability of F can be obtained by applying Propositions 2.1 and 2.5 to the Doi-Hopf datum $(H^{\text{op}} \otimes H, \underline{A}^2, C)$.

4.1. Frobenius properties for Yetter-Drinfeld modules.

Proposition 4.2. *Let H be a quasi-Hopf algebra, let C be an H -bimodule coalgebra and let A be an H -bicomodule algebra. The forgetful functor $F : \mathcal{YD}(H)_A^C \rightarrow \mathcal{M}_A$ is Frobenius if and only if C is finite dimensional and there exists $t = u_i \otimes c_i \in A \otimes C$ such that*

$$uu_i \otimes c_i = u_i u_{[0] \langle 0 \rangle} \otimes S(u_{[-1]}) c_i u_{[0] \langle 1 \rangle},$$

for all $u \in A$, and the map $\kappa : A \otimes C \rightarrow A \otimes C$,

$$\begin{aligned} \kappa(u \otimes c) &= \langle *c, \tilde{X}_\lambda^1 S(\theta_1^1(\tilde{X}_\lambda^2)_1 \mathbf{p}^1) f^2 \cdot (c_i)_{\underline{2}} \cdot \theta_{(1)_2}^2(\tilde{x}_\rho^2)_2 p^2 S(\theta^3 \tilde{x}_\rho^3) \rangle \\ &\quad uu_i \theta_{(0)}^2 \tilde{x}_\rho^1(\tilde{X}_\lambda^3)_{\langle 0 \rangle} \otimes S(\theta_2^1(\tilde{X}_\lambda^2)_2 \mathbf{p}^2) f^1 \cdot (c_i)_{\underline{1}} \cdot \theta_{(1)_1}^2(\tilde{x}_\rho^2)_1 p^1(\tilde{X}_\lambda^3)_{\langle 1 \rangle}, \end{aligned}$$

is an isomorphism. As before, $f = f^1 \otimes f^2$ is the Drinfeld's twist, $p_L = \mathbf{p}^1 \otimes \mathbf{p}^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3$ and $p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3)$. Consequently, F is a Frobenius functor if C is a Frobenius coalgebra in ${}_H \mathcal{M}_H$.

Proof. Recall that $p_R^{H^{\text{op}}} = q_R^H = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2$. Therefore $p_R^{H^{\text{op}} \otimes H} = (q^1 \otimes p^1) \otimes (q^2 \otimes p^2)$. Moreover, p_R and q_R satisfy the property

$$(4.3) \quad p^1 h \otimes p^2 = h_{(1,1)} p^1 \otimes h_{(1,2)} p^2 S(h_2), \quad h q^1 \otimes q^2 = q^1 h_{(1,1)} \otimes S^{-1}(h_2) q^2 h_{(1,2)},$$

for all $h \in H$. This allows us to compute that

$$\begin{aligned} &\tilde{x}_{\underline{\rho}^2}^1 \otimes (\tilde{x}_{\underline{\rho}^2}^2)_1 (q^1 \otimes p^1) \otimes (\tilde{x}_{\underline{\rho}^2}^2)_2 (q^2 \otimes p^2) (S^{-1} \otimes S) (\tilde{x}_{\underline{\rho}^2}^3) \\ &= \theta_{(0)}^2 \tilde{x}_\rho^1(\tilde{X}_\lambda^3)_{\langle 0 \rangle} \otimes \left(q^1 g_1^1 S(\theta^1 \tilde{X}_\lambda^2)_1 \otimes \theta_{(1)_1}^2(\tilde{x}_\rho^2)_1(\tilde{X}_\lambda^3)_{\langle 1 \rangle_{(1,1)}} p^1 \right) \\ &\quad \otimes \left(\tilde{X}_\lambda^1 S^{-1}(g^2) q^2 g_2^1 S(\theta^1 \tilde{X}_\lambda^2)_2 \otimes \theta_{(1)_2}^2(\tilde{x}_\rho^2)_2(\tilde{X}_\lambda^3)_{\langle 1 \rangle_{(1,2)}} p^2 S(\theta^3 \tilde{x}_\rho^3(\tilde{X}_\lambda^3)_{\langle 1 \rangle_2}) \right) \\ &\stackrel{(4.3)}{=} \theta_{(0)}^2 \tilde{x}_\rho^1(\tilde{X}_\lambda^3)_{\langle 0 \rangle} \otimes \left(q^1 g_1^1 G^1 S(\theta_2^1(\tilde{X}_\lambda^2)_2) f^1 \otimes \theta_{(1)_1}^2(\tilde{x}_\rho^2)_1 p^1(\tilde{X}_\lambda^3)_{\langle 1 \rangle} \right) \\ &\stackrel{(1.17)}{=} \left(\tilde{X}_\lambda^1 S^{-1}(g^2) q^2 g_2^1 G^2 S(\theta_1^1(\tilde{X}_\lambda^2)_1) f^2 \otimes \theta_{(1)_2}^2(\tilde{x}_\rho^2)_2 p^2 S(\theta^3 \tilde{x}_\rho^3) \right) \\ &\stackrel{(*)}{=} \theta_{(0)}^2 \tilde{x}_\rho^1(\tilde{X}_\lambda^3)_{\langle 0 \rangle} \otimes \left(S(\theta_2^1(\tilde{X}_\lambda^2)_2 \mathbf{p}^2) f^1 \otimes \theta_{(1)_1}^2(\tilde{x}_\rho^2)_1 p^1(\tilde{X}_\lambda^3)_{\langle 1 \rangle} \right) \\ &\quad \otimes \left(\tilde{X}_\lambda^1 S(\theta_1^1(\tilde{X}_\lambda^2)_1 \mathbf{p}^1) f^2 \otimes \theta_{(1)_2}^2(\tilde{x}_\rho^2)_2 p^2 S(\theta^3 \tilde{x}_\rho^3) \right), \end{aligned}$$

in $A \otimes (H^{\text{op}} \otimes H)^{\otimes 2}$. Here $f^{-1} = G^1 \otimes G^2$ is a second copy of f^{-1} . At $(*)$, we used the equality

$$(4.4) \quad q^1 g_1^1 G^1 \otimes S^{-1}(g^2) q^2 g_2^1 G^2 = S(\mathbf{p}^2) \otimes S(\mathbf{p}^1),$$

see [5, (4.13)]. The first part in the statement is an immediate consequence of Proposition 2.1, applied to $(H^{\text{op}} \otimes H, \underline{A}^2, C)$. The second part can be deduced easily from Proposition 2.3, using the monoidal category isomorphism ${}_H\mathcal{M}_H \cong \mathcal{M}_{H^{\text{op}} \otimes H}$. \square

The natural question arises whether the converse of the final statement in Proposition 4.2 holds: is C a Frobenius coalgebra in ${}_H\mathcal{M}_H$ if F is Frobenius? Theorem 2.4 provides an answer to questions of this type, but, unfortunately, it cannot be applied in our situation. We have an algebra map $\varepsilon : H \rightarrow k$, but the associated map $\tilde{\varepsilon} : H \rightarrow H^{\text{op}} \otimes H$, $\tilde{\varepsilon}(h) = S(h_1) \otimes h_2$, is not surjective. However, in the case where $C = H$, we obtain an affirmative answer to the question, using the structure theorem for quasi-Hopf algebras. We need some preliminary results first.

Lemma 4.3. *Let H be a finite dimensional quasi-Hopf algebra and let μ be its modular element. H_μ is the vector space equipped with left and right H -action and right H -coaction ρ given by the formulas*

$$h \cdot \bar{h} \cdot h' = \mu(h_1)h_2\bar{h}h' ; \quad \rho(\bar{h}) = \mu(x^1)x^2\bar{h}_1 \otimes x^3\bar{h}_2.$$

H_μ is a quasi-Hopf H -bimodule. If λ is a non-zero left cointegral on H then

$$\zeta : H_\mu \rightarrow {}^*H, \quad \xi(\bar{h}) = S(\bar{h}) \dashrightarrow \lambda$$

is an isomorphism in ${}_H\mathcal{M}_H^H$, with inverse given by the formula

$$\zeta^{-1}({}^*h) = \langle {}^*h, S^{-1}(q^1t_1p^1) \rangle S^{-1}(q^2t_2p^2),$$

where t is a left integral in H such that $\lambda(S^{-1}(t)) = 1$.

Proof. Recall from (3.16) that ${}^*H \in {}_H\mathcal{M}_H^H$. It follows from the comments made before Definition 5.3 in [18] that H_μ is a quasi-Hopf H -bimodule and that ζ is an isomorphism in ${}_H\mathcal{M}_H^H$. Our present contribution is the explicit description of ζ^{-1} . Using (4.3) and (3.17), one can verify that ζ and ζ^{-1} are inverses. \square

Let H be a quasi-bialgebra. \mathcal{YD}_H^H will be a short notation for $\mathcal{YD}(H)_H^H$.

Lemma 4.4. *Let H be a quasi-Hopf algebra. If the forgetful functor $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is Frobenius then H is finite dimensional and there exists an element $t \in H$ such that*

$$(4.5) \quad \mu(h_{(2,1)})S(h_1)th_{(2,2)} = \mu(h)t,$$

*for all $h \in H$, where μ is the modular element of H . Furthermore, the map $\Upsilon : {}^*H \rightarrow H$,*

$$(4.6) \quad \Upsilon({}^*h) = \langle \mu, y_1^2x^1X_1^3 \rangle \langle {}^*h, X^1S(y_1^1X_1^2\mathbf{p}^1)f^2t_2y_{(2,2)}^2x_2^2p^2S(y^3x^3) \rangle S(y_2^1X_2^2\mathbf{p}^2)f^1t_1y_{(2,1)}^2x_1^2p^1X_2^3$$

is an isomorphism and satisfies

$$(4.7) \quad \langle \mu, h_{(2,1)} \rangle S(h_1)\Upsilon({}^*h)h_{(2,2)} = \langle \mu, h_{(2,1)} \rangle \Upsilon(S(h_{(2,2)}) \dashrightarrow {}^*h \leftarrow h_1),$$

*for all ${}^*h \in {}^*H$ and $h \in H$.*

Proof. It follows from Proposition 1.3 that H is finite dimensional if F is Frobenius. Then the forgetful functor $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H \rightarrow \mathcal{M}_H$ is also Frobenius. Applying the first part of Theorem 2.4 to the algebra map $\mu : H \rightarrow k$, we find $t \in H$ obeying $\mu(h_{(2,1)})S(h_1)th_{(2,2)} = \mu(h)t$, for all $h \in H$, and such that $\Upsilon : {}^*H \rightarrow H$,

$$\Upsilon({}^*h) = \langle \mu, y_1^2x^1X_1^3 \rangle \langle {}^*h, X^1S(y_1^1X_1^2\mathbf{p}^1)f^2t_2y_{(2,2)}^2x_2^2p^2S(y^3x^3) \rangle S(y_2^1X_2^2\mathbf{p}^2)f^1t_1y_{(2,1)}^2x_1^2p^1X_2^3$$

is an isomorphism. Note that we made also use of the computation performed in the proof of Proposition 4.2, applied to the Doi-Hopf datum $(H^{\text{op}} \otimes H, \underline{H}^2, H)$.

In order to prove (4.7), we compute that

$$\begin{aligned} \langle \mu, h_{(2,1)} \rangle S(h_1)\Upsilon({}^*h)h_{(2,2)} &= \langle \mu, y_1^2x^1(X^3h_2)_1 \rangle \langle {}^*h, X^1S(y_1^1X_1^2\mathbf{p}^1)f^2t_2y_{(2,2)}^2x_2^2p^2S(y^3x^3) \rangle \\ &\quad S(y_2^1X_2^2\mathbf{p}^2h_1)f^1t_1y_{(2,1)}^2x_1^2p^1(X^3h_2)_2 \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.23,1.12)}{=} \langle \mu, y_1^2 x^1 h_{(2,2)_1} X_1^3 \rangle \langle *h, h_1 X^1 S((y^1 h_{(2,1)})_1 X_1^2 \mathbf{p}^1) f^2 t_2 y_{(2,2)}^2 x_2^2 p^2 S(y^3 x^3) \rangle \\
& \quad S((y^1 h_{(2,1)})_2 X_2^2 \mathbf{p}^2) f^1 t_1 y_{(2,1)}^2 x_1^2 p^1 h_{(2,2)_2} X_2^3 \\
& \stackrel{(4.3,1.12)}{=} \langle *h, h_1 X^1 S((y^1 (h_2)_1)_1 X_1^2 \mathbf{p}^1) f^2 t_2 (y^2 (h_2)_{(2,1)})_{(2,2)} x_2^2 p^2 S(y^3 (h_2)_{(2,2)} x^3) \rangle \\
& \quad \mu((y^2 (h_2)_{(2,1)})_1 x^1 X_1^3) S((y^1 (h_2)_1)_2 X_2^2 \mathbf{p}^2) f^1 t_1 (y^2 (h_2)_{(2,1)})_{(2,1)} x_1^2 p^1 X_2^3 \\
& \stackrel{(1.12,1.17)}{=} \langle *h, h_1 X^1 S(y_1^1 X_1^2 \mathbf{p}^1) f^2 (S(h_{(2,1)_1}) t h_{(2,1)}_{(2,2)})_2 y_{(2,2)}^2 x_2^2 p^2 S(h_{(2,2)} y^3 x^3) \rangle \\
& \quad \langle \mu, h_{(2,1)}_{(2,1)} y_1^2 x^1 X_1^3 \rangle S(y_2^1 X_2^2 \mathbf{p}^2) f^1 (S(h_{(2,1)_1}) t h_{(2,1)}_{(2,2)})_1 y_{(2,1)}^2 x_1^2 p^1 X_2^3 \\
& \stackrel{(4.5)}{=} \langle \mu, h_{(2,1)} y_1^2 x^1 X_1^3 \rangle \langle *h, h_1 X^1 S(y_1^1 X_1^2 \mathbf{p}^1) f^2 t_2 y_{(2,2)}^2 x_2^2 p^2 S(h_{(2,2)} y^3 x^3) \rangle \\
& \quad S(y_2^1 X_2^2 \mathbf{p}^2) f^1 t_1 y_{(2,1)}^2 x_1^2 p^1 X_2^3 \\
& = \langle \mu, h_{(2,1)} \rangle \Upsilon(S(h_{(2,2)}) \dashv *h \dashv h_1),
\end{aligned}$$

for all $h \in H$ and $*h \in *H$, finishing the proof of the Lemma. \square

We will need the following formulas, see [18], and [5, (4.14)].

$$(4.8) \quad [1 \otimes S^{-1}(h)]V = [h_2 \otimes 1]V\Delta(S^{-1}(h_1)),$$

$$(4.9) \quad q_R = [\tilde{q}^2 \otimes 1]V\Delta(S^{-1}(\tilde{q}^1)),$$

$$(4.10) \quad p_R = \Delta(S(\tilde{p}^1))U[\tilde{p}^2 \otimes 1], \text{ and}$$

$$(4.11) \quad S(g^1)q^1 g_1^2 \otimes q^2 g_2^2 = S(p^2)f^1 \otimes S(p^1)f^2.$$

By the definitions of p_R and p_L and using (1.13) we obtain that

$$(4.12) \quad X_1^1 p^1 \otimes X_2^1 p^2 S(X^2) \otimes X^3 = x^1 \otimes x^2 S(x_1^3 \mathbf{p}^1) \otimes x_2^3 \mathbf{p}^2 \text{ and}$$

$$(4.13) \quad y^1 \mathbf{p}^1 \otimes y^2 \mathbf{p}_1^2 \otimes y^3 \mathbf{p}_2^2 = X_1^2 \mathbf{p}^1 S^{-1}(X^1) \otimes X_2^2 \mathbf{p}^2 \otimes X^3.$$

Theorem 4.5 generalizes [16, Theorem 4.2] to the quasi-Hopf algebra setting. Note that our approach is different from the one in [16].

Theorem 4.5. *For a quasi-Hopf algebra H , the following assertions are equivalent:*

- (i) *The forgetful functor $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is Frobenius;*
- (ii) *H is finite dimensional and unimodular;*
- (iii) *H is finite dimensional and Frobenius as a coalgebra in ${}_H\mathcal{M}_H$.*

Proof. (ii) \Leftrightarrow (iii). If H is finite dimensional then H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular, see the proof of (iv) of Theorem 3.4.

(i) \Rightarrow (ii). It follows from Lemma 4.4 that H is finite dimensional, and that there exists $t \in H$ satisfying (4.5) and such that $\Upsilon : *H \rightarrow H$ defined in (4.6) is an isomorphism. For all $*h \in *H$, we have that

$$\begin{aligned}
\Upsilon(*h) & \stackrel{(1.17,4.3)}{=} \langle \mu, y_1^2 x^1 X_1^3 \rangle \langle *h, X^1 S(\mathbf{p}^1) f^2 (S(y^1 X^2) t y_2^2 x^2 X_{(2,1)}^3)_2 p^2 S(y^3 x^3 X_{(2,2)}^3) \rangle \\
& \quad S(\mathbf{p}^2) f^1 (S(y^1 X^2) t y_2^2 x^2 X_{(2,1)}^3)_1 p^1 \\
& \stackrel{(4.4,1.12)}{=} \langle \mu, y_1^2 X_{(1,1)}^3 x^1 \rangle \langle *h, S^{-1}(g^2 S(X^1)) q^2 (g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2)_2 p^2 S(y^3 X_2^3 x^3) \rangle \\
& \quad q^1 (g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2)_1 p^1,
\end{aligned}$$

Let $\zeta : H \rightarrow *H$ be the isomorphism defined in Lemma 4.3. For all $h \in H$, we have that

$$\begin{aligned}
(\Upsilon \circ \zeta)(h) & = \langle \mu, y_1^2 X_{(1,1)}^3 x^1 \rangle \langle \lambda, S^{-1}(g^2 S(X^1)) q^2 (g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2)_2 p^2 S(h y^3 X_2^3 x^3) \rangle \\
& \quad q^1 (g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2)_1 p^1 \\
& \stackrel{(4.9,4.10)}{=} \langle \lambda, S^{-1}(g^2 S(X^1)) V^2 (S^{-1}(q^1) g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2 S(\mathbf{p}^1))_2 U^2 S(h y^3 X_2^3 x^3) \rangle \\
& \quad \langle \mu, y_1^2 X_{(1,1)}^3 x^1 \rangle q^2 V^1 (S^{-1}(q^1) g^1 S(y^1 X^2) t y_2^2 X_{(1,2)}^3 x^2 S(\mathbf{p}^1))_1 U^1 \mathbf{p}^2
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(3.23,4.8)}{=} \langle \lambda, S^{-1}(\mathbf{q}^1 g_1^2 S(X^1)_1) g^1 S(y^1 X^2) \mathbf{t} y_2^2 X_{(1,2)}^3 x^2 S(z^2 h_1 y_1^3 X_{(2,1)}^3 x_1^3 \mathbf{p}^1) \rangle \\
& \stackrel{(3.14)}{=} \langle \mu, z^1 y_1^2 X_{(1,1)}^3 x^1 \rangle \mathbf{q}^2 g_2^2 S(X^1)_2 z^3 h_2 y_2^3 X_{(2,2)}^3 x_2^3 \mathbf{p}^2 \\
& \stackrel{(1.17,4.11)}{=} \langle \lambda, S^{-1}(f^1) X_2^1 p^2 S(y^1 X^2) \mathbf{t} y_2^2 X_{(1,2)}^3 x^2 S(z^2 h_1 y_1^3 X_{(2,1)}^3 x_1^3 \mathbf{p}^1) \rangle \\
& \quad \langle \mu, z^1 y_1^2 X_{(1,1)}^3 x^1 \rangle S(X_1^1 p^1) f^2 z^3 h_2 y_2^3 X_{(2,2)}^3 x_2^3 \mathbf{p}^2,
\end{aligned}$$

and

$$\begin{aligned}
& \langle \mu, h_1 \rangle (\Upsilon \circ \zeta)(h_2) \stackrel{(1.12)}{=} \langle \lambda, S^{-1}(f^1) X_2^1 p^2 S(y^1 X^2) \mathbf{t} y_2^2 X_{(1,2)}^3 x^2 S(h_{(1,2)} z^2 y_1^3 X_{(2,1)}^3 x_1^3 \mathbf{p}^1) \rangle \\
& \quad \langle \mu, z^1 h_{(1,1)} y_1^2 X_{(1,1)}^3 x^1 \rangle S(X_1^1 p^1) f^2 h_2 z^3 y_2^3 X_{(2,2)}^3 x_2^3 \mathbf{p}^2 \\
& \stackrel{(3.18,4.5)}{=} \langle \lambda, S^{-1}(f^1 h_1) X_2^1 p^2 S(z^1 y^1 X^2) \mathbf{t} (z_2^1 y^2)_2 X_{(1,2)}^3 x^2 S(z^2 y_1^3 X_{(2,1)}^3 x_1^3 \mathbf{p}^1) \rangle \\
& \quad \langle \mu, (z_2^1 y^2)_1 X_{(1,1)}^3 x^1 \rangle S(X_1^1 p^1) f^2 h_2 z^3 y_2^3 X_{(2,2)}^3 x_2^3 \mathbf{p}^2 \\
& \stackrel{(1.13,1.12)}{=} \langle \lambda, S^{-1}(f^1 h_1) X_2^1 p^2 S(z^1 y^1 X^2) \mathbf{t} z_2^2 y_{(1,2)}^2 X_{(1,1)_2}^3 w_1^2 x^2 S(z^3 y_2^3 X_{(1,2)}^3 w^2 x_1^3 \mathbf{p}^1) \rangle \\
& \quad \langle \mu, z_1^2 y_{(1,1)}^2 X_{(1,1)_1}^3 w_1^1 x^1 \rangle S(X_1^1 p^1) f^2 h_2 y^3 X_2^3 w^3 x_2^3 \mathbf{p}^2 \\
& \stackrel{(4.12)}{=} \langle \lambda, S^{-1}(f^1 h_1) X_2^1 p^2 S(z^1 y^1 X^2) \mathbf{t} z_2^2 (y^2 X_1^3)_{(1,2)} P^2 S(z^3 (y^2 X_1^3)_2) \rangle \\
& \quad \langle \mu, z_1^2 (y^2 X_1^3)_{(1,1)} P^1 \rangle S(X_1^1 p^1) f^2 h_2 y^3 X_2^3 \\
& \stackrel{(4.3)}{=} \langle \lambda, S^{-1}(f^1 h_1) X_2^1 p^2 S(z^1 y^1 X^2) \mathbf{t} z_2^2 P^2 S(z^3) \rangle \\
& \quad \langle \mu, z_1^2 P^1 y^2 X_1^3 \rangle S(X_1^1 p^1) f^2 h_2 y^3 X_2^3 \\
& \stackrel{(4.12,1.12)}{=} \langle \lambda, S^{-1}(f^1 h_1) x^2 S(z^1 x_{(1,1)}^3 y^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S(z^3) \rangle \\
& \quad \langle \mu, z_1^2 P^1 x_{(1,2)}^3 y^2 \mathbf{p}_1^2 \rangle S(x^1) f^2 h_2 x_2^3 y^3 \mathbf{p}_2^2,
\end{aligned}$$

where $p_R = P^1 \otimes P^2$ a second copy of p_R . Taking $h = \alpha$, we find that

$$\begin{aligned}
& \langle \mu, \alpha_1 \rangle (\Upsilon \circ \zeta)(\alpha_2) \stackrel{(1.20)}{=} \langle \lambda, S^{-1}(\gamma^1) x^2 S(z^1 x_{(1,1)}^3 y^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S(z^3) \rangle \langle \mu, z_1^2 P^1 x_{(1,2)}^3 y^2 \mathbf{p}_1^2 \rangle S(x^1) \gamma^2 x_2^3 y^3 \mathbf{p}_2^2 \\
& \stackrel{(1.18,3.18)}{=} \langle \lambda, S^{-1}(\alpha) w^1 X^2 x^2 S(z^1 x_{(1,1)}^3 y^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S((w^2 X_1^3)_2 z^3) \rangle \\
& \quad \langle \mu, (w^2 X_1^3)_1 z_1^2 P^1 x_{(1,2)}^3 y^2 \mathbf{p}_1^2 \rangle S(X^1 x^1) \alpha w^3 X_2^3 x_2^3 y^3 \mathbf{p}_2^2 \\
& \stackrel{(4.5,1.12)}{=} \langle \lambda, S^{-1}(\alpha) w^1 X^2 x^2 S(z^1 (w^2 X_1^3)_1 x_{(1,1)}^3 y^1 \mathbf{p}^1) \mathbf{t} z_2^2 (w^2 X_1^3)_{(2,1)_2} P^2 S(z^3 (w^2 X_1^3)_{(2,2)}) \rangle \\
& \quad \langle \mu, z_1^2 (w^2 X_1^3)_{(2,1)_1} P^1 x_{(1,2)}^3 y^2 \mathbf{p}_1^2 \rangle S(X^1 x^1) \alpha w^3 X_2^3 x_2^3 y^3 \mathbf{p}_2^2 \\
& \stackrel{(4.3)}{=} \langle \lambda, S^{-1}(\alpha) w^1 S(z^1 w_1^2 y^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S(z^3) \rangle \langle \mu, z_1^2 P^1 w_2^2 y^2 \mathbf{p}_1^2 \rangle \alpha w^3 y^3 \mathbf{p}_2^2 \\
& \stackrel{(4.13)}{=} \langle \mu, z_1^2 P^1 \mathbf{p}^2 \rangle \langle \lambda, S^{-1}(\alpha) S(z^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S(z^3) \rangle \alpha = x\alpha,
\end{aligned}$$

with

$$(4.14) \quad x = \langle \mu, z_1^2 P^1 \mathbf{p}^2 \rangle \langle \lambda(S^{-1}(\alpha) S(z^1 \mathbf{p}^1) \mathbf{t} z_2^2 P^2 S(z^3)) \rangle \in k.$$

Let $\hbar = (\Upsilon \circ \zeta)^{-1}(\alpha)$. Since $\mu(\alpha_1)(\Upsilon \circ \zeta)(\alpha_2) = x\alpha = x\Upsilon\zeta(\hbar)$, we have that $\mu(\alpha_1)\alpha_2 = x\hbar$, and therefore $0 \neq \mu(\alpha) = x\varepsilon(\hbar)$, and $x \neq 0$. We have now all the ingredients to prove that H is unimodular, that is, $\mu = \varepsilon$. Observe that

$$\begin{aligned}
(\varepsilon \circ \Upsilon)(^*h) &= \langle \mu, y_1^2 x^1 X^3 \rangle \langle ^*h, X^1 S(y^1 X^2 S^{-1}(\beta)) \mathbf{t} y_2^2 x^2 \beta S(y^3 x^3) \rangle \\
&= \langle \mu, y_1^2 p^1 \mathbf{p}^2 \rangle \langle ^*h, S(y^1 \mathbf{p}^1) \mathbf{t} y_2^2 p^2 S(y^3) \rangle,
\end{aligned}$$

for all $^*h \in ^*H$. Thus $(\varepsilon \circ \Upsilon)(\lambda \leftarrow S^{-1}(\alpha)) = x$. Applying ε to (4.7) we obtain that

$$\begin{aligned}
\langle \mu, h \rangle (\varepsilon \circ \Upsilon)(^*h) &= \langle \mu, h_{(2,1)} \rangle (\varepsilon \circ \Upsilon)(S(h_{(2,2)}) \leftarrow ^*h \leftarrow h_1) \\
&= \langle \mu, h_{(2,1)} y_1^2 p^1 \mathbf{p}^2 \rangle \langle ^*h, h_1 S(y^1 \mathbf{p}^1) \mathbf{t} y_2^2 p^2 S(h_{(2,2)} y^3) \rangle
\end{aligned}$$

for all $^*h \in ^*H$ and $h \in H$. Taking $^*h = \lambda \leftarrow S^{-1}(\alpha) = \langle \mu, \alpha_1 \rangle S(\alpha_2) \leftarrow \lambda$, we conclude that

$$\langle \mu, h \rangle x = \langle \mu, h \rangle (\varepsilon \circ \Upsilon)(\lambda \leftarrow S^{-1}(\alpha)) = \langle \mu, (\alpha h_2)_1 y_1^2 p^1 \mathbf{p}^2 \rangle \langle \lambda, h_1 S(y^1 \mathbf{p}^1) \mathbf{t} y_2^2 p^2 S((\alpha h_2)_2 y^3) \rangle$$

$$\stackrel{(3.18)}{=} \langle \mu, y_1^2 p^1 \mathfrak{p}^2 \rangle \langle \lambda, S^{-1}(\alpha h_2) h_1 S(y^1 \mathfrak{p}^1) t y_2^2 p^2 S(y^3) \rangle = \varepsilon(h)x,$$

for all $h \in H$, and it follows that $\mu = \varepsilon$ since $x \neq 0$.

(ii) \Rightarrow (i). If H is finite dimensional and unimodular, then H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, see the proof of (iv) of Theorem 3.4. By Proposition 2.3, the forgetful functor $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H \rightarrow \mathcal{M}_H$ is Frobenius, and therefore F is a Frobenius functor as well since $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$ and \mathcal{YD}_H^H are isomorphic. \square

4.2. Separability for the category of Yetter-Drinfeld modules. We now focus on the separability of the forgetful functor $F : \mathcal{YD}(H)_A^C \rightarrow \mathcal{M}_A$. Applying Proposition 2.5 and using the isomorphism between the categories $\mathcal{YD}(H)_A^C$ and $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{A}^2}^C$, we obtain necessary and sufficient conditions for the separability of F .

Proposition 4.6. *Let H be a quasi-Hopf algebra, let A be an H -bicomodule algebra, and let C be a finite dimensional H -bimodule coalgebra with dual basis $c_j \otimes c^j \in C \otimes {}^*C$. Then the forgetful functor $F : \mathcal{YD}(H)_A^C \rightarrow \mathcal{M}_A$ is separable if and only if there exists a k -linear map $\bar{\Lambda} : C \rightarrow A \otimes {}^*C$, $\bar{\Lambda}(c) = \bar{\Lambda}^1(c) \otimes \bar{\Lambda}^2(c) \in A \otimes {}^*C$, such that*

$$\begin{aligned} & \bar{\Lambda}^1(c) u_{[0]_{(0)}} \otimes S(u_{[0]_{(1)}}) \bar{\Lambda}^2(c) \leftarrow u_{[-1]} \\ &= u_{[0]_{(0)}} \bar{\Lambda}^1(S(u_{[-1]}) \cdot c \cdot u_{[0]_{(1)}}) \otimes \bar{\Lambda}^2(S(u_{[-1]}) \cdot c \cdot u_{[0]_{(1)}}), \\ & (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \Theta_{(0)}^2 \bar{\Lambda}^1(S(\tilde{x}_\lambda^2 \Theta^1) f^1 \cdot c_1 \cdot (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \Theta_{(1)}^2) \otimes \bar{\Lambda}^2(S(\tilde{x}_\lambda^2 \Theta^1) f^1 \cdot c_1 \cdot (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \Theta_{(1)}^2) \\ & \otimes S(\tilde{x}_\lambda^1) f^2 \cdot c_2 \cdot (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3 = \bar{\Lambda}^1(c) (\tilde{x}_\lambda^3 \tilde{q}_\rho^1 \Theta_{(0)}^2)_{(0)} \tilde{x}_\rho^1 \otimes \left(g^1 S(\Theta_{(1)}^2 \tilde{x}_\rho^3) \tilde{q}_\rho^2 \Theta^3 \rightarrow c^j \right. \\ & \leftarrow S(\tilde{x}_\lambda^1) \mathfrak{q}^1 (\tilde{x}_\lambda^2)_1 \Theta_1^1 \right) \left(g^2 S((\tilde{x}_\lambda^3 \tilde{q}_\rho^1 \Theta_{(0)}^2)_{(1)} \tilde{x}_\rho^2) \bar{\Lambda}^2(c) \leftarrow \mathfrak{q}^2 (\tilde{x}_\lambda^2)_2 \Theta_2^1 \right) \otimes c_i, \\ & \bar{\Lambda}^2(S(\tilde{x}_\lambda^1) f^2 \cdot c_2 \cdot (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3) \left(S(\tilde{x}_\lambda^2) \Theta^1 \bar{\Lambda}^1(S(\tilde{x}_\lambda^1) f^2 \cdot c_2 \cdot (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3)_{[-1]} \theta^1 \tilde{p}_\lambda^1 \right) f^1 \\ & \cdot c_1 \cdot (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \Theta_{(1)}^2 \bar{\Lambda}^1(S(\tilde{x}_\lambda^1) f^2 \cdot c_2 \cdot (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3)_{[0]_{(1)}} \theta_{(1)}^2 \tilde{p}_\rho^2 S(\theta^3) \Big) \\ & (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \Theta_{(0)}^2 \bar{\Lambda}^1(S(\tilde{x}_\lambda^1) f^2 \cdot c_2 \cdot (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3)_{[0]_{(0)}} \theta_{(0)}^2 \tilde{p}_\rho^1 \tilde{p}_\lambda^2 = \varepsilon_C(c) 1_A, \end{aligned}$$

for all $c \in C$ and $u \in A$. \rightarrow and \leftarrow are the left and right H -actions on *C induced by the H -bimodule structure of C , namely $\langle h \rightarrow {}^*c \leftarrow h', c \rangle = \langle {}^*c, h' \cdot c \cdot h \rangle$.

Proof. The follows from Proposition 2.5 (vii) after we show that

$$\begin{aligned} & (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \Theta_{(0)}^2 \otimes (S^{-1}(F^2) \otimes g^1) (S^{-1} \otimes S) \left((p^2 \otimes q^2) (S(\tilde{x}_\lambda^2 \Theta^1)_2 f_2^1 \otimes (\tilde{x}_\lambda^3)_{(1)_{(1,2)}} (\tilde{X}_\rho^2)_2 \Theta_{(1,2)}^2) \right. \\ & \left. (S(\tilde{x}_\lambda^1) f^2 \otimes (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3) \right) \otimes (S^{-1}(F^2) \otimes g^2) (S^{-1} \otimes S) ((p^1 \otimes q^1) \\ & (S(\tilde{x}_\lambda^2 \Theta^1)_1 f_1^1 \otimes (\tilde{x}_\lambda^3)_{(1)_{(1,1)}} (\tilde{X}_\rho^2)_1 \Theta_{(1,1)}^2)) = (\tilde{x}_\lambda^3 \tilde{q}_\rho^1 \Theta_{(0)}^2)_{(0)} \tilde{x}_\rho^1 \otimes (S(\tilde{x}_\lambda^1) \mathfrak{q}^1 (\tilde{x}_\lambda^2)_1 \Theta_1^1 \\ & \otimes g^1 S(\Theta_{(1)}^2 \tilde{x}_\rho^3) \tilde{q}_\rho^2 \Theta^3) \otimes \left(\mathfrak{q}^2 (\tilde{x}_\lambda^2)_2 \Theta_2^1 \otimes g^2 S((\tilde{x}_\lambda^3 \tilde{q}_\rho^1 \Theta_{(0)}^2)_{(1)} \tilde{x}_\rho^2) \right) \end{aligned}$$

in $A \otimes (H^{\text{op}} \otimes H)^{\otimes 2}$. Note that the Drinfeld twist in H^{op} is $S^{-1}(g^2) \otimes S^{-1}(g^1)$, where $g^1 \otimes g^2$ is the inverse of the Drinfeld twist f in H . The above equality follows from a straightforward computation using (1.17), the axioms of a quasi-Hopf algebra and of a bicomodule algebra over a quasi-bialgebra, and the formula

$$(4.15) \quad f^2 S^{-1}(F^2 f_2^1 p^2) \otimes F^1 f_1^1 p^1 = \mathfrak{q}^1 \otimes S(\mathfrak{q}^2),$$

which is a consequence of (1.21) and of the fact that $f^1 \beta S(f^2) = S(\alpha)$. We leave the verification of the details to the reader. \square

The conditions in Proposition 4.6 are rather complicated. However they are fulfilled if C is a coseparable coalgebra in ${}_H\mathcal{M}_H$, see Proposition 2.6. We will show that the converse is also true in the particular case when $C = A = H$. It will turn out that the structure theorem for quasi-Hopf algebras plays a crucial role in the proof, indicating that it is not possible to make a generalization for arbitrary H -bimodule coalgebras C . We also emphasise the fact that this result is also new in the case where H is a classical Hopf algebra. We first need some preparatory results.

Lemma 4.7. *Let H be a finite dimensional quasi-Hopf algebra and let λ be a left cointegral on H . Then the following relations hold, for all $h, h' \in H$,*

$$(4.16) \quad \langle \lambda, q^2 h_2 U^2 \rangle q^1 h_1 U^1 = \langle \mu, x^1 \rangle \langle \lambda, S^{-1}(f^1) h S(x^2) \rangle f^2 x^3,$$

$$(4.17) \quad \langle \lambda, h' h_2 U^2 \rangle h_1 U^1 = \langle \mu, x^1 \rangle \langle \lambda, S^{-1}(f^1) h'_2 p^2 h S(x^2) \rangle S(h'_1 p^1) f^2 x^3.$$

Proof.

$$\begin{aligned} \langle \lambda, q^2 h_2 U^2 \rangle q^1 h_1 U^1 &\stackrel{(4.15)}{=} \langle \lambda, S^{-1}(F^1 f_1^1 p^1) h_2 U^2 \rangle f^2 S^{-1}(F^2 f_2^1 p^2) h_1 U^1 \\ &\stackrel{(1.17)}{=} \langle \lambda, V^2(S^{-1}(f^1) h)_2 U^2 \rangle f^2 V^1(S^{-1}(f^1) h)_1 U^1 \\ &\stackrel{(3.14)}{=} \langle \mu, x^1 \rangle \langle \lambda, S^{-1}(f^1) h S(x^2) \rangle f^2 x^3; \\ \langle \lambda, h' h_2 U^2 \rangle h_1 U^1 &\stackrel{(2.23)}{=} \langle \lambda, h' q^2 p_2^2 h_2 U^2 \rangle S(p^1) q^1 p_1^2 h_1 U^1 \\ &\stackrel{(2.22)}{=} \langle \lambda, q^2 (h'_2 p^2 h)_2 U^2 \rangle S(h'_1 p^1) q^2 (h'_2 p^2 h)_1 U^1 \\ &\stackrel{(4.16)}{=} \langle \mu, x^1 \rangle \langle \lambda, S^{-1}(f^1) h'_2 p^2 h S(x^2) \rangle S(h'_1 p^1) f^2 x^3, \end{aligned}$$

as stated. \square

Lemma 4.8. *Let H be a finite dimensional quasi-Hopf algebra and let μ be the modular element of H . Consider the isomorphism $\zeta : H_\mu \rightarrow {}^*H$ from Lemma 4.3. We can define Yetter-Drinfeld module structures on H_μ and *H such that ζ is an isomorphism of Yetter-Drinfeld modules. These structures are given by (4.21, 4.22) and (4.18, 4.20).*

Proof. The first aim is to make *H into a Yetter-Drinfeld module. This goes in several steps.

1) By Proposition 2.5 (v) we know that $H \otimes {}^*H \in \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$ via the structure given by $(\hbar \otimes {}^*h) \cdot h = \hbar h_{(2,1)} \otimes S(h_{(2,2)}) \rightharpoonup^* h \leftharpoonup h_1$ and

$$\begin{aligned} \rho(\hbar \otimes {}^*h) &= \hbar x_1^3 X^1 Y_1^2 \otimes \left(g^1 S(q^2(x_{(2,1)}^3 X^2 Y_2^2)_2) x_{(2,2)}^3 X^3 Y^3 \rightharpoonup^* h^i \leftharpoonup S(x^1) f^2 S^{-1}(F^2 S(x^2 Y^1)_2 f_2^1 p^2) \right) \\ &\quad \left(g^2 S(q^1(x_{(2,1)}^3 X^2 Y_2^2)_1) \rightharpoonup^* h \leftharpoonup S^{-1}(F^1 S(x^2 Y^1)_1 f_1^1 p^1) \right) \otimes h_i \\ &\stackrel{(1.17)}{=} \hbar x_1^3 X^1 Y_1^2 \otimes \left(g^1 S(q^2 X_2^2 Y_{(2,2)}^2) X^3 Y^3 \rightharpoonup^* h^i \leftharpoonup S(x^1) q^1 x_1^2 Y_1^1 \right) \\ &\stackrel{(4.15)}{=} \left(g^2 S(x_2^3 q^1 X_1^2 Y_{(2,1)}^2) \rightharpoonup^* h \leftharpoonup q^2 x_2^2 Y_2^1 \right) \otimes h_i, \end{aligned}$$

for all $h, \hbar \in H$ and ${}^*h \in {}^*H$.

2) $H \otimes {}^*H$ is a left H -module via $h \cdot (\hbar \otimes {}^*h) = \hbar \hbar \otimes {}^*h$. Consequently $M \otimes_H (H \otimes {}^*H) \in \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$, for every $M \in \mathcal{M}_H$, and the fact that $M \otimes {}^*H$ is naturally isomorphic to $M \otimes_H (H \otimes {}^*H)$, entails that $M \otimes {}^*H \in \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$. The structure maps are the following:

$$\begin{aligned} (m \otimes {}^*h) \cdot h &= m h_{(2,1)} \otimes S(h_{(2,2)}) \rightharpoonup^* h \leftharpoonup h_1, \\ \rho(m \otimes {}^*h) &= m x_1^3 X^1 Y_1^2 \otimes \left(g^1 S(q^2 X_2^2 Y_{(2,2)}^2) X^3 Y^3 \rightharpoonup^* h^i \leftharpoonup S(x^1) q^1 x_1^2 Y_1^1 \right) \\ &\quad \left(g^2 S(x_2^3 q^1 X_1^2 Y_{(2,1)}^2) \rightharpoonup^* h \leftharpoonup q^2 x_2^2 Y_2^1 \right) \otimes h_i, \end{aligned}$$

for all $m \in M$, $h \in H$ and $*h \in *H$.

3) $k \in \mathcal{M}_H$ by restriction of scalars via ε , hence $k \otimes *H \cong *H \in \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$, with structure maps

$$(4.18) \quad *h \triangleleft h = S(h_2) \rightharpoonup *h \leftharpoonup h_1$$

$$(4.19) \quad \rho(*h) = (g^1 S(q^2 Y_2^2) Y^3 \rightharpoonup h^i \leftharpoonup S(x^1) \mathfrak{q}^1 x_1^2 Y_1^1) (g^2 S(x^3 q^1 Y_1^2) \rightharpoonup *h \leftharpoonup \mathfrak{q}^2 x_2^2 Y_2^1) \otimes h_i,$$

for all $*h \in *H$ and $h \in H$.

4) We have an isomorphism of categories $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$, see (4.1), and therefore $*H \in \mathcal{YD}_H^H$.

At this point, the proof is basically finished: ζ can be used to transport the Yetter-Drinfeld structure on $*H$ to H_μ . It remains to compute the explicit structure maps. The right H -action on $*H$ is (4.18). The right H -coaction can be computed from the action and coaction (4.18-4.19) on $*H \in \mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$ using (4.2). Using the fact that $*H$ is an algebra in ${}_H\mathcal{M}_H$ and the equations (1.12), (4.3) and (2.23), we find that this coaction is given by the formula

$$(4.20) \quad \rho(*h) = (g^1 S(q^2 Y_2^2) Y^3 \rightharpoonup h^i \leftharpoonup X^1 Y_1^1) (g^2 S(X^3 q^1 Y_1^2) \rightharpoonup *h \leftharpoonup X^2 Y_2^1) \otimes h_i,$$

where $h_i \otimes h^i \in H \otimes *H$ is the finite dual basis of H .

Let us describe the structure on H_μ . The H -bimodule structure on H_μ (see Lemma 4.3) induces the following right H -module structure on H_μ :

$$(4.21) \quad h \triangleleft h = \mu(S(h_1)_1) S(h_1)_2 h h_2,$$

for all $h, h \in H$. The right H -coaction $\rho : H_\mu \rightarrow H_\mu \otimes H$ is given by the formula

$$(4.22) \quad \rho(h) = h_{(0)} \otimes h_{(1)} := \mu(S(Y_2^2 \mathfrak{p}^2 X^1)_1 f_1^1 x^1) S(Y_2^2 \mathfrak{p}^2 X^1)_2 f_2^1 x^2 h_1 Y_1^3 X^2 \otimes Y^1 S(Y_1^2 \mathfrak{p}^1) f^2 x^3 h_2 Y_2^3 X^3,$$

for all $h \in H$. The proof is finished after we show that ζ is right H -linear and colinear with respect to the structures (4.21, 4.22) and (4.18, 4.20). Since $\zeta : H_\mu \rightarrow *H$ is an H -bimodule map, it follows immediately that it is also right H -linear. For all $h \in H$ we have that

$$\begin{aligned} \rho(\zeta(h)) &\stackrel{(4.20)}{=} (g^1 S(q^2 Y_2^2) Y^3 \rightharpoonup h^i \leftharpoonup X^1 Y_1^1) (g^2 S(h X^3 q^1 Y_1^2) \rightharpoonup \lambda \leftharpoonup X^2 Y_2^1) \otimes h_i \\ &\stackrel{(4.3)}{=} \langle \lambda, X^2 (Y^1 h_i S(X_1^3 Y^2))_2 U^2 S(h) \rangle h^i \otimes X^1 (Y^1 h_i S(X_1^3 Y^2))_1 U^2 X_2^3 Y^3 \\ &\stackrel{(3.23)}{=} \langle \lambda, X^2 (Y^1 h_i S(h_1 X_1^3 Y^2))_2 U^2 \rangle h^i \otimes X^1 (Y^1 h_i S(h_1 X_1^3 Y^2))_1 U^2 h_2 X_2^3 Y^3 \\ &\stackrel{(4.17)}{=} \langle \mu, x^1 \rangle \langle \lambda, S^{-1}(f^1) X_2^2 \mathfrak{p}^2 Y^1 h_i S(x^2 h_1 X_1^3 Y^2) \rangle h^i \otimes X^1 S(X_1^2 \mathfrak{p}^1) f^2 x^3 h_2 X_2^3 Y^3 \\ &= \langle \mu, x^1 \rangle S(x^2 h_1 X_1^3 Y^2) \rightharpoonup \lambda \leftharpoonup S^{-1}(f^1) X_2^2 \mathfrak{p}^2 Y^1 \otimes X^1 S(X_1^2 \mathfrak{p}^1) f^2 x^3 h_2 X_2^3 Y^3 \\ &\stackrel{(3.18)}{=} \langle \mu, S(X_2^2 \mathfrak{p}^2 Y^1)_1 f_1^1 x^1 \rangle S(X_2^2 \mathfrak{p}^2 Y^1)_2 f_2^1 x^2 h_1 X_1^3 Y^2 \rightharpoonup \lambda \otimes X^1 S(X_1^2 \mathfrak{p}^1) f^2 x^3 h_2 X_2^3 Y^3 \\ &\stackrel{(4.22)}{=} \zeta(h_{(0)}) \otimes h_{(1)}, \end{aligned}$$

proving that ζ is right H -colinear. \square

Theorem 4.9. *For a finite dimensional quasi-Hopf algebra H , the following assertions are equivalent:*

- (i) *The forgetful functor $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is separable;*
- (ii) *H is coseparable as a coalgebra in ${}_H\mathcal{M}_H$;*
- (iii) *H is unimodular and cosemisimple.*

Proof. (ii) \Rightarrow (i). If H is a coseparable coalgebra in ${}_H\mathcal{M}_H$, then the forgetful functor $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H \rightarrow \mathcal{M}_H$ is separable, by Proposition 2.6. It follows that F is separable since the categories \mathcal{YD}_H^H and $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$ are isomorphic.

(i) \Rightarrow (iii). If $F : \mathcal{YD}_H^H \rightarrow \mathcal{M}_H$ is separable, then the forgetful functor $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H \rightarrow \mathcal{M}_H$ is separable, and it follows from Proposition 2.5 (v) that there exists a left H -linear morphism $\Lambda : H \otimes H \rightarrow H \otimes^* H$, $\Lambda(1 \otimes h) = \bar{\Lambda}^1(h) \otimes \bar{\Lambda}^2(h) \in H \otimes^* H$, that is also a morphism in $\mathcal{M}(H^{\text{op}} \otimes H)_{\underline{H}^2}^H$ and, a fortiori, in \mathcal{YD}_H^H , satisfying

$$(4.23) \quad \begin{aligned} \varepsilon(h)1 &= \bar{\Lambda}^2(S(x^1)f^2h_2x_{(2,2)}^3X^3Y^3) \left(S(x^2Y^1\bar{\Lambda}^1(S(x^1)f^2h_2x_{(2,2)}^3X^3Y^3)_1y^1\mathfrak{p}^1)f^1h_1x_{(2,1)}^3X^2Y_2^2 \right. \\ &\quad \left. \times \bar{\Lambda}^1(S(x^1)f^2h_2x_{(2,2)}^3X^3Y^3)_{(2,2)}y_2^2p^2S(y^3) \right) x_1^3X^1Y_1^2\bar{\Lambda}^1(S(x^1)f^2h_2x_{(2,2)}^3X^3Y^3)_{(2,1)}y_1^2p^1\mathfrak{p}^2, \end{aligned}$$

for all $h \in H$. For any $M \in \mathcal{M}_H$, the map $\Lambda_M : M \otimes H \rightarrow M \otimes^* H$ given by

$$\Lambda_M(m \otimes h) = m\bar{\Lambda}^1(h) \otimes \bar{\Lambda}^2(h),$$

for all $m \in M$ and $h \in H$, is a morphism in \mathcal{YD}_H^H . The Yetter-Drinfeld structure on $M \otimes^* H$ is described explicitly in the proof Lemma 4.8; the structure on $M \otimes H$ is similar, and is given by the formulas

$$\begin{aligned} (m \otimes h) \cdot h &= m \cdot h_{(2,1)} \otimes S(h_1)hh_{(2,2)}; \\ \rho(m \otimes h) &= m \cdot x_1^3X^1Y_1^2(\mathfrak{p}_1^2)_{(2,1)} \otimes S(x^2Y^1(\mathfrak{p}_1^2)_1)f^1h_1x_{(2,1)}^3X^2Y_2^2(\mathfrak{p}_1^2)_{(2,2)} \\ &\quad \otimes S(x^1\mathfrak{p}^1)f^2h_2x_{(2,2)}^3X^3Y^3\mathfrak{p}_2^2 \in M \otimes H \otimes H, \end{aligned}$$

for all $m \in M$ and $h, h \in H$. Taking $M = k$ as a right H -module by restriction of scalars via ε , we obtain a morphism $\Lambda_k : H \rightarrow^* H$, $\Lambda_k(h) = \varepsilon(\bar{\Lambda}^1(h))\bar{\Lambda}^2(h)$, in \mathcal{YD}_H^H . The structure of *H in \mathcal{YD}_H^H is given by the formulas (4.18, 4.20). $H \in \mathcal{YD}_H^H$ with structure

$$h \triangleleft h = S(h_1)hh_2; \quad \rho(h) = S(x^2Y^1\mathfrak{p}_{(1,1)}^2)f^1h_1x_1^3Y^2\mathfrak{p}_{(1,2)}^2 \otimes S(x^1\mathfrak{p}^1)f^2h_2x_2^3Y^3\mathfrak{p}_2^2.$$

Recall from Lemma 4.8 that we have an isomorphism $\zeta : H_\mu \rightarrow^* H$ in \mathcal{YD}_H^H , and consider the composition $\chi = \zeta^{-1} \circ \Lambda_k : H \rightarrow H_\mu$ in \mathcal{YD}_H^H . The right H -linearity of χ is expressed by the formula

$$\chi(S(h_1)hh_2) = \mu(S(h_1)_1)S(h_1)_2\chi(h)h_2.$$

Taking $h = \alpha$, we find that $\varepsilon(h)\chi(\alpha) = \mu(S(h_1)_1)S(h_1)_2\chi(\alpha)h_2$, for all $h \in H$. Applying ε to both sides of this equation, we find that

$$(4.24) \quad \varepsilon(h)\vartheta(\alpha) = \mu(S(h))\vartheta(\alpha),$$

for all $h \in H$, with $\vartheta = \varepsilon \circ \chi$. The right H -colinearity of χ comes out as

$$\begin{aligned} &\chi(S(x^2Y^1\mathfrak{p}_{(1,1)}^2)f^1h_1x_1^3Y^2\mathfrak{p}_{(1,2)}^2) \otimes S(x^1\mathfrak{p}^1)f^2h_2x_2^3Y^3\mathfrak{p}_2^2 \\ &= \mu(S(Y_2^2\mathfrak{p}^2X^1)_1f_1^1x^1)S(Y_2^2\mathfrak{p}^2X^1)_2f_2^1x^2\chi(h)_1Y_1^3X^2 \otimes Y^1S(Y_1^2\mathfrak{p}^1)f^2x^3\chi(h)_2Y_2^3X^3, \end{aligned}$$

for all $h \in H$. Applying $\varepsilon \otimes H$ to this equality we obtain that

$$(4.25) \quad \mu(S(Y_2^2\mathfrak{p}^2)f^1)Y^1S(Y_1^2\mathfrak{p}^1)f^2\chi(h)Y^3 = \vartheta(S(x^2Y^1\mathfrak{p}_{(1,1)}^2)f^1h_1x_1^3Y^2\mathfrak{p}_{(1,2)}^2)S(x^1\mathfrak{p}^1)f^2h_2x_2^3Y^3\mathfrak{p}_2^2,$$

for all $h \in H$. Let $h = S(\mathfrak{q}_1^2)\mathfrak{h}\mathfrak{q}_2^2$ in (4.25), and multiply both sides of it to the left by \mathfrak{q}^1 . Using the formulas $Y_1^2\mathfrak{p}^1S^{-1}(Y^1) \otimes Y_2^2\mathfrak{p}^2 \otimes Y^3 = y^1\mathfrak{p}^1 \otimes y^2\mathfrak{p}_1^2 \otimes y^3\mathfrak{p}_2^2$, (1.12), (1.17) and (2.23) we deduce that

$$\mu(S(y^2)f^1)S(y^1)f^2\chi(h)y^3 = \vartheta(S(x^2Y^1)f^1h_1x_1^3Y^2)S(x^1)f^2h_2x_2^3Y^3,$$

for all $h \in H$, or, equivalently,

$$\chi(h) = \mu(g^1S(Z^2))\vartheta(S(x^2Y^1)f^1h_1x_1^3Y^2)g^2S(x^1Z^1)f^2h_2x_2^3Y^3Z^3.$$

In particular,

$$\chi(\alpha) \stackrel{(1.20)}{=} \mu(g^1S(Z^2))\vartheta(S(x^2Y^1)\gamma^1x_1^3Y^2)g^2S(x^1Z^1)\gamma^2x_2^3Y^3Z^3 \stackrel{(1.18)}{=} \mu(g^1S(Z^2))g^2S(Z^1)\alpha Z^3,$$

which implies that $\vartheta(\alpha) = \varepsilon\chi(\alpha) = \varepsilon(\alpha) \neq 0$. Therefore, (4.24) is equivalent to $\varepsilon(h) = \mu(S(h))$, for all $h \in H$, and this is clearly equivalent to $\mu = \varepsilon$. Thus we have shown that H is unimodular, and, in particular, that $\chi(\alpha) = \alpha$. Applying ε to (4.23) we find that

$$\bar{\Lambda}^2(S(x^1)f^2h_2x_2^3Y^3)(\beta S(x^2Y^1\bar{\Lambda}^1(S(x^1)f^2h_2x_2^3Y^3)_1p^1)f^1h_1x_1^3Y^2\bar{\Lambda}^1(S(x^1)f^2h_2x_2^3Y^3)_2p^2) = \varepsilon(h),$$

for all $h \in H$. Take $h = \alpha$; using (1.20) and (1.18), a similar computation shows that

$$\varepsilon(\alpha) = \bar{\Lambda}^2(\alpha)(\beta S(\bar{\Lambda}^1(\alpha)_1p^1)\alpha\bar{\Lambda}^1(\alpha)_2p^2) = \varepsilon(\bar{\Lambda}^1(\alpha))\bar{\Lambda}^2(\alpha)(\beta S(p^1)\alpha p^2) \stackrel{(1.16)}{=} \Lambda_k(\alpha)(\beta).$$

Now $\Lambda_k = \zeta\chi$ and $\chi(\alpha) = \alpha$, so $0 \neq \varepsilon(\alpha) = \Lambda_k(\alpha)(\beta) = \zeta(\alpha)(\beta) = \lambda(\beta S(\alpha)) \stackrel{(3.19)}{=} \lambda(S^{-1}(\alpha)\beta)$, which implies that H is cosemisimple.

(ii) \Leftrightarrow (iii) follows from Proposition 3.6. \square

Remark 4.10. Replacing H by H^{op} , we obtain necessary and sufficient conditions for the Frobenius property and separability of the forgetful functor $F : {}_H\mathcal{YD}^H \rightarrow {}_H\mathcal{M}$. Unimodularity (resp. unimodularity and cosemisimplicity) of H and H^{op} are equivalent, and therefore $F : {}_H\mathcal{YD}^H \rightarrow {}_H\mathcal{M}$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple). If H is finite dimensional, then ${}_H\mathcal{YD}^H \cong_{D(H)}\mathcal{M}$, where $D(H)$ is the quantum double of H , see [17]. So our results imply that the algebra extension $H \hookrightarrow D(H)$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple).

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