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Abstract

Subspace identification methods may produce unreliable model estimates when a small number of noisy measurements are available. In such cases, the accuracy of the estimated parameters can be improved by using prior knowledge about the system. The prior knowledge considered in this paper is constraints on the impulse response. It is motivated by availability of information about the steady-state gain, overshoot, and rise time of the system, which in turn can be expressed as constraints on the impulse response. The method proposed has two steps: 1) estimation of the impulse response with linear equality and inequality constraints, and 2) realization of the estimated impulse response. The problem on step 1 is shown to be a convex quadratic programming problem. In the case of prior knowledge expressed as equality constraints, the problem on step 1 admits a closed form solution. In the general case of equality and inequality constraints, the solution is computed by standard numerical optimization methods. We illustrate the performance of the method on a mass-spring-damper system.

Keywords: system identification, subspace methods, prior knowledge, behavioral approach.

1 Introduction

The main goal of this paper is to improve the efficiency of standard subspace algorithms when the user has prior information about the process to be identified. This information may be obtained from the laws of physics governing the system, preliminary experiments such as a step response or a response to a sinusoidal input signal, or from an expert knowledge. For example, the user may know the steady-state gain, the settling time, or the dominant time constant of the system. The developed identification method is a generic modeling tool and is not limited to a specific applications area. Indeed, *any* application can benefit from exploiting prior knowledge in the identification process, provided that 1) such prior knowledge is available and 2) there is a method that can use it.

Prior knowledge about a system can be expressed naturally as constraints on its behavior, *e.g.*, overshoot and rise time are defined in terms of the step response [14]. In parametric identification, however, the model is represented by a parameter vector—coefficients of a transfer function or a state space representation. The identification problem then becomes a parameter estimation problem and inclusion of the prior knowledge requires its re-formulation as constraints on the parameter vector. This may be nontrivial and leads to more complicated optimization problems. Indeed, linear constraints on the system's behavior often result in nonlinear constraints on the parameter vector [17].

When we deal with subspace identification, it is difficult to introduce such prior knowledge directly into the model structure. Subspace identification does not resort to an explicit cost functions and uses a state space representation of the system that is known up to a similarity transformation. Thus, introducing physically meaningful prior information into a state space model to-be-estimated by a subspace identification algorithm seems to be a challenging problem.

In this paper, we bypass the difficulties related to inclusion of prior knowledge in parameter estimation by the following a two-step method:

1. estimation of the impulse response, and

2. realization of the estimated impulse response.

The prior knowledge is imposed on the estimated impulse response in step 1. The method is based on a result from [11], where it is shown that, for exact data, the impulse response of a linear time-invariant system can be computed directly from data by solving an overdetermined system of linear equations. In case of noisy data, generically the system has no solution. Then, a heuristic subspace approach is to estimate the impulse response by solving the system approximately in the least squares sense. We refer to this approach as *data-driven impulse response estimation*.

As shown in the paper, imposing prior knowledge in the method of [11] leads to a convex quadratic programming problem, for which fast and efficient methods exist (the active-set methods [9, Chapter 23], [3, Chapter 5], [5] and the interior point methods [15, 4]). In the case of prior knowledge expressed as equality constraints, the data-driven impulse response estimation problem is a constrained least squares problem and admits a closed form solution (see Section 4.1).

Once the impulse response is estimated, computing the system's parameters of a state-space representation is a classical realization problem [6, 21]. We use Kung's algorithm [8], which involves computation of the singular value decomposition (SVD) and solution of a least squares problem. The overall method requires solution of a least squares problem (in case of equality constraints) or a convex quadratic programming problem (in case of inequality constraints), an SVD, and another least-squares problem. The overall computational cost of the method is comparable to that of classical subspace methods.

In the context of subspace system identification, prior knowledge about stability and passivity of the model is considered in [10, 7]. The approach used in [10, 7] consists in including a regularization term in the least-squares cost function for the estimation of the model parameters. The main result is that, for sufficiently large values of the regularization parameter, the identified model is stable and passive. More recently subspace identification with prior knowledge was considered in [18, 2, 1]. In [2, 1], prior knowledge about the steady-state gain of the system is taken into account by a modified PO-MOESP algorithm, where a constrained least squares problem is solved. The approach of [18] generalizes the regularization approach of [10, 7] to a Bayesian framework for including prior knowledge about the parameter vector. The method however involves a solution of a nonconvex optimization problem, which makes it comparable to the prediction error methods.

The two-step method proposed in this paper is similar to the method of Alenany *et al.* [1]. The latter is also based on estimation of the impulse response by a subspace algorithm, however, it involves a truncation of an infinite sum, which results in approximate impulse response estimate. In contrast, the method proposed here yields exact results when the data is exact under standard identifiability assumptions: persistency of excitation and controllability (Section 3).

Other advantages of the method proposed are the computational cost and simplicity of implementation. The least squares problem in our approach (see (3)) is smaller dimensional than the least squares problem in the method of Alenani *et al.*, see equation (21) in [1]. This is due to the lower block triangular Toeplitz structure of the estimated parameter matrix. In order to take into account the structure Alenani *et al.* vectorize the system of equations. This is at the price of multiplication of the problem dimensions and complicated (Kronecker) structure of the resulting coefficients matrices. In contrast, equation (3) is an unstructured least squares problem that can be solved without vectorization. Our method is implemented in Matlab and is publicly available. The core part of the code is based on the formula of the analytical solution (6) and is listed in the appendix. The implementation of the method of Alenany for general equality constraint (equation (28) in [1]) seems nontrivial. The authors informed us that it is currently implemented only for prior knowledge in the form of a steady-state gain.

2 Preliminaries and notation

The set of real numbers is denoted by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real-valued matrices. $0_{m\times n}$ denotes the $m \times n$ zero matrix and A^+ denotes the Moore-Penrose pseudoinverse of the matrix A. The set of infinite vector-valued time series with q variables is denoted by $(\mathbb{R}^q)^{\mathbb{N}}$.

We use the behavioral notion of a dynamical system. A discrete-time dynamical system $\mathscr{B}$ with q variables is a subset of the signal space $(\mathbb{R}^q)^{\mathbb{N}}$, see [16]. The notation $\mathscr{B}|_{[t_1,t_2]}$ stands for the restriction of the behavior on the interval $[t_1,t_2]$, *i.e.*,

$$\mathscr{B}|_{[t_1,t_2]} := \{\, w \in (\mathbb{R}^q)^{t_2-t_1} \mid \text{ there are } w_\mathrm{p} \text{ and } w_\mathrm{f}, \text{ such that } w_\mathrm{p} \wedge w \wedge w_\mathrm{f} \in \mathscr{B} \,\},$$

where $w_\mathrm{p} \wedge w$ is the concatenation of the trajectories w_p and w.

We assume that a trajectory w of $\mathscr{B}$ has an input/output partition $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, with $\mathtt{m}$ inputs and $\mathtt{p}$ outputs, where $q = \mathtt{p} + \mathtt{m}$. In general, a permutation Πw of the variables is needed in order to have all inputs as the first variables.

A finite dimensional linear time-invariant system $\mathscr{B}$ is a closed shift-invariant subspace of $(\mathbb{R}^q)^{\mathbb{N}}$. The input/state/output representation of a linear time-invariant system $\mathscr{B}$ is denoted by

$$\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D) := \{\, (u,y) \mid \text{there is } x \text{ such that } \dot{x} = Ax + Bu \text{ and } y = Cx + Du \,\}.$$

The *order* of the system is the smallest state dimension $\mathtt{n} = \operatorname{row\,dim}(x)$. The *lag* of a linear time-invariant system is the observability index of the system [13].

The Hankel matrix with t block rows, composed of the sequence $w \in (\mathbb{R}^q)^T$ is denoted by

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-t+1) \\ w(2) & w(3) & \cdots & w(T-t+2) \\ w(3) & w(4) & \cdots & w(T-t+3) \\ \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & \cdots & w(T) \end{bmatrix}.$$

The time series $u = \big(u(1),\dots,u(T)\big)$ is persistently exciting of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank.

In general, quantities marked with bar (*e.g.*, $\bar{h}$) refer to noise free or true values. Quantities marked with hat (*e.g.*, $\widehat{h}$) refer to estimates of the true values, and quantities marked with tilde (*e.g.*, $\widetilde{h}$) refer to estimation errors, or perturbation on the true value due to noise.

3 Impulse response estimation

In this section, we consider the problem of computing the first $\eta + 1$ samples $\big(h(0), h(1), \dots, h(\eta)\big)$ of the impulse response

$$h = \big(h(0), h(1), h(2), \dots\big), \qquad \text{where} \quad h(t) \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$$

of a linear time-invariant system $\mathscr{B}$ from a finite trajectory

$$w_\mathrm{d} := \big(w_\mathrm{d}(1), \dots, w_\mathrm{d}(T)\big)$$

of the system.

Lemma 1 ([20])**.** *Let $\mathscr{B}$ be a linear time-invariant system of order $\mathtt{n}$ with lag ℓ. Under the following assumptions:*

1. *the data w_d is exact,* i.e., $w_d \in \mathscr{B}$*,*

2. *the system is $\mathscr{B}$ is controllable, and*

3. *the input component u_d of the trajectory w_d is persistently exciting of order $\mathtt{n}+\ell+1$,*

the image of the Hankel matrix $\mathscr{H}_t(w_d)$ is the space $\mathscr{B}|_{[1,t]}$ of all t-samples long trajectories of the system $\mathscr{B}$, i.e.,

$$\text{image}\,\big(\mathscr{H}_t(w_d)\big) = \mathscr{B}|_{[1,t]}.$$

Let e_i be the i-th unit vector, δ be discrete-time unit pulse, and h_i be the i-th column of the impulse response h. Then,

$$\underbrace{(0,\dots,0)}_{\ell} \wedge (e_i\delta, h_i), \qquad \text{for} \quad i = 1,\dots,\mathtt{m},$$

are trajectories of $\mathscr{B}$. The prefix of ℓ leading zero samples to the trajectories $(e_i\delta, h_i)$ fixes the zero initial conditions, see [12, Lemma 1]. Then, by Lemma 1, there exist vectors $g_1,\dots,g_{\mathtt{m}}$, such that

$$\mathscr{H}_{\ell+\eta+1}(w_{\mathrm{d}})g_i = \begin{bmatrix} 0_{q\ell\times 1} \\ \begin{bmatrix} e_i \\ h_i(0) \end{bmatrix} \\ \begin{bmatrix} 0_{\mathtt{m}\times 1} \\ h_i(1) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 0_{\mathtt{m}\times 1} \\ h_i(\eta) \end{bmatrix} \end{bmatrix}, \qquad \text{for} \quad i = 1,\dots,\mathtt{m}. \tag{1}$$

Selecting the block-rows in (1) that correspond to the h_i's in the right-hand-side of the equations, we have

$$\underbrace{\mathscr{H}_{\eta+1}(y_{\mathrm{d}})}_{Y_{\mathrm{f}}} g_i = \underbrace{\begin{bmatrix} h_i(0) \\ h_i(1) \\ \vdots \\ h_i(\eta) \end{bmatrix}}_{H_i}, \qquad \text{for } i = 1,\dots,\mathtt{m}$$

or with

$$G := \begin{bmatrix} g_1 & \cdots & g_{\mathtt{m}} \end{bmatrix} \qquad \text{and} \qquad H := \begin{bmatrix} H_1 & \cdots & H_{\mathtt{m}} \end{bmatrix},$$

we have the matrix equation

$$Y_{\mathrm{f}} G = H. \tag{2}$$

Therefore, the problem of computing the first $\eta+1$ samples of the impulse response h from the data w_{d} reduces to the one of finding the matrix G in (2).

We aim to compute the unknown impulse response samples $h_i(0), h_i(1), \dots, h_i(\eta)$ in the right-hand-side of (1). Let V be the matrix formed by the block-rows of the right-hand-side of (1) that are specified and $\mathscr{A}$ be the matrix formed by the corresponding block-rows of the left-hand-side in (1). We obtain a system of linear equations for G

$$\mathscr{A} G = V. \tag{3}$$

Under the conditions of Lemma 1, (3) has a solution. Moreover, by (2), for any solution G of (3), the matrix $Y_\mathrm{f} G$ contains the first $\eta + 1$ samples of the impulse response of $\mathscr{B}$. Equations (3) and (2) lead to an algorithm (see Algorithm 1) for the computation of the impulse response from a general trajectory of the system.

Algorithm 1 Impulse response estimation. `uy2h`

Input: Trajectory w_d, system lag ℓ, and number of impulse response samples η.

1: Solve the system of equations (3) and let G be the computed solution.

2: Let $H = Y_\mathrm{f} G$.

Output: H.

Note 2 (Parameters ℓ and η.). The algorithm has two user defined parameters:

- the lag ℓ, which is a natural number reflecting prior knowledge about the model complexity, and
- the number $\eta + 1$ of the estimated samples of the impulse response.

The parameters ℓ and η can be chosen independently, however, Assumption 3 of Lemma 1 imposes an upper bound on them.

Note 3 (Recursive computation of the impulse response). Algorithm 1 finds the first $\eta + 1$ samples of the impulse response; however, the persistency of excitation condition imposes a limitation on how big η can be. This limitation can be avoided by a modification of the algorithm that computes iteratively overlapping blocks of $\ell + 1$ consecutive samples of h and reconstructs the full sequence $h = \big(h(0), h(1), h(2), \dots\big)$ from them.

4 Imposing prior knowledge

Next, we consider prior knowledge about the impulse response in the form of equality

$$EH = F \tag{4}$$

and inequality

$$E'H \leq F'$$

constraints. For example with $E = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$, EH is an approximation of the steady-state gain of the system, and with

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{i \times (\eta+1)}, \tag{5}$$

EH contains the first i samples of the step response of the system.

4.1 Impulse response estimation with equality constraints

All solutions of (4) are a particular solution, *e.g.*, the least-norm solution $(EY_\mathrm{f})^+F$, plus a vector in the null space of EY_f. Therefore, G is of the form

$$G = (EY_\mathrm{f})^+F + NZ, \quad \text{for some } Z,$$

where the columns of N form a basis for the null space of EY_f. From (3), we have

$$\mathscr{A}\big((EY_\mathrm{f})^+F + NZ\big) = V.$$

Therefore,

$$Z = (\mathscr{A}N)^+\big(V - \mathscr{A}(EY_\mathrm{f})^+F\big).$$

Finally, for the impulse response, we have

$$H = Y_\mathrm{f}\left((EY_\mathrm{f})^+F + N(\mathscr{A}N)^+\left(V - \mathscr{A}(EY_\mathrm{f})^+F\right)\right). \tag{6}$$

4.2 Impulse response estimation with equality and inequality constraints

In the case of equality and inequality constraints, the problem is a quadratic program

$$\begin{aligned} &\text{minimize} \quad \text{over } g \quad \|\mathscr{A}G - V\| \\ &\text{subject to} \quad EY_\mathrm{f}G = F \quad \text{and} \quad E'Y_\mathrm{f}G \leq F'. \end{aligned} \tag{7}$$

It does not admit an analytical solution but due to convexity it can be solved globally and efficiently. We use an active-set algorithm [5], which is implemented in the function `lsqlin` of Matlab. The resulting method for impulse response estimation with linear equality/inequality constraints is summarized in Algorithm 2.

4.3 Impulse response realization $h \mapsto (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$

After the estimation of the impulse response h from the input/output data w, we obtain a state space representation $\mathscr{B}(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ of the identified model, using Kung's method [8]. The resulting method proposed is outlined in Algorithm 3. A MATLAB implementation of the method is available from:

Algorithm 2 Impulse response estimation with linear equality/inequality constraints. `uy2h_pk`

Input: Trajectory w_d, system lag ℓ, equality (E,F) and inequality (E',F') constraints, and number of impulse response samples η.

1: **if** there are equality constraints only **then**
2: Compute H via (6).
3: **else**
4: Solve the quadratic program (7).
5: Let $H = Y_\mathrm{f} G$
6: **end if**

Output: H.

Algorithm 3 Subspace identification with prior knowledge on the impulse response. `uy2ss_pk`

Input: Trajectory w_d, equality (E,F) and inequality (E',F') prior knowledge, and system order $\mathtt{n}$.

1: $(w_\mathrm{d},(E,F),(E',F')) \mapsto \widehat{H}$ using Algorithm 2, compute an estimate $\widehat{h}$ of the impulse response.
2: $(\widehat{h},\mathtt{n}) \mapsto (\widehat{A},\widehat{B},\widehat{C},\widehat{D})$ using Kung's method, compute the parameters of a state space representation of the system.

Output: Identified model $\mathscr{B}_{\mathrm{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$.

```
http://homepages.vub.ac.be/~imarkovs/software/detss_pk.tar
```

Note 4. The final model identified by Algorithm 3 may not satisfy the constraints due to approximate realization on step 2. In case of noisy data, the estimated impulse response $\widehat{h}$ on step 1 of Algorithm 3 is generically not exactly realizable by a linear time-invariant system with order less than or equal to $\mathtt{n}$. Then, step 2 involves an approximation. The impulse response of the model $\widehat{\mathscr{B}} := \mathscr{B}_{\mathrm{i/s/o}}(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$ is as a result generically not equal to $\widehat{h}$ and may not satisfy the prior knowledge. In Section 5, we show the effect of Step 2 on the constraint satisfaction. The empirical study shows that although the constraints are not exactly satisfied they are much better satisfied by the approximated model computed by Algorithm 3 than they are by models computed by alternative methods that do not use the prior knowledge. Moreover, taking into account the constraints on step one leads to a better estimation of the true data generating system (measured in the $\mathscr{H}_2$-norm sense).

5 Numerical experiments

5.1 Simulation setup

In the numerical examples we use a mass-spring-damper system

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + d\frac{\mathrm{d}}{\mathrm{d}t}y + ky = u,$$

where the model parameters are chosen as $m = 1$, $d = 1$, and $k = 10$. The data is regularly sampled with a sampling period 0.2sec from the continuous-time system. We denote with $\bar{\mathscr{B}}$ by discretized true data generating model. The identification data w_d is obtained in the errors-in-variables setting:

$$w_\mathrm{d} = \bar{w} + \widetilde{w}, \quad \text{where } \bar{w} \in \bar{\mathscr{B}}, \text{ and } \widetilde{w} \text{ is zero mean,white Gaussian process with covariance } s^2 I.$$

Here $\bar{w}$ is the "true value" of the trajectory w_d. The input $\bar{u}$ is a zero mean, white, random process with uniform distribution in the interval $[0,1]$. The data consists of $T = 50$ samples and the noise standard deviation is $s = 0.01$.

5.2 A single equality constraint

In this subsection, we consider that prior knowledge is in the form of a single equality constraint (4), with $E := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ and $F := E\bar{H}$, with $\bar{H}$ being the matrix of the first $\eta + 1$ impulse response samples of the data generating system $\bar{\mathscr{B}}$. Note that $EH = \sum_{t=0}^{\eta} h(t)$ is the η-th sample of the step response and for $\eta \to \infty$, EH converges to the steady-state gain of the system. Therefore, for large η, the equality constraint $EH = F$ can be used to impose prior knowledge of the steady-state gain.

Using the data w_d and the prior knowledge specified by the pair (E, F), we estimate the model $\widehat{\mathscr{B}}$ with Algorithm 3 (`uy2ss_pk`). For comparison, we also estimate models using two alternative methods:

- `uy2ss` — the two-stage method that is not using the prior knowledge, and
- `n4sid` — the N4SID method [19], implemented in the Identification Toolbox of Matlab, called with the default parameters.

The experiment is repeated $N = 100$ times with different noise realizations (Monte-Carlo simulation). Let $\widehat{\mathscr{B}}^{(k)}$ be the identified model in the kth repetition of the experiment and let $\widehat{h}^{(k)}$ be the estimated impulse response. Denote with $\|\mathscr{B}\|$ the 2-norm of the system $\mathscr{B}$. We compare the average relative estimation errors

$$\varepsilon_{\mathscr{B}} = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{\mathscr{B}} - \widehat{\mathscr{B}}^{(k)}\|}{\|\bar{\mathscr{B}}\|} \qquad \text{and} \qquad \varepsilon_h = \frac{1}{N}\sum_{k=1}^{N} \frac{\|\bar{H} - \widehat{H}^{(k)}\|}{\|\bar{H}\|_\mathrm{F}}$$

of, respectively, the identified system and the identified first $\eta + 1 = 10$ samples of the impulse response. In addition, we show the satisfaction of the constraint $E\widehat{H} = F$ by the Frobenius norm of the residual errors

$$\varepsilon'_{\mathscr{B}} = \frac{1}{N}\sum_{k=1}^{N} \|E\widehat{H}^{(k)}_{\mathscr{B}} - F\|_\mathrm{F} \qquad \text{and} \qquad \varepsilon'_h = \frac{1}{N}\sum_{k=1}^{N} \|E\widehat{H}^{(k)} - F\|_\mathrm{F},$$

where $\widehat{H}^{(k)}_{\mathscr{B}}$ is the impulse response of the identified model $\widehat{\mathscr{B}}^{(k)}$. As pointed out in Note 4, with noisy data, in general, $\widehat{H}^{(k)} \neq \widehat{H}^{(k)}_{\mathscr{B}}$ due to approximation in the computation of the state space realization of the model.

The results are reported in Table 1. In the performance measures $\varepsilon_{\mathscr{B}}$ and ε_h, the proposed Algorithm 1 (`uy2ss_pk`) improves the results of Algorithm 2 (`uy2ss`). However, using a single equality constraints, in this simulation example `n4sid` produces a better result than `uy2ss_pk`. In the next section, we show that with two or more equality constraints `uy2ss_pk` outperforms `n4sid`.

The equality constraints are satisfied exactly by the estimated impulse response obtained with Algorithm 1 but not by the estimated impulse response obtained with Algorithm 2. Furthermore, the estimated model of Algorithm 3 (`uy2ss_pk`) approximates the equality constraint better than the alternative methods (`uy2ss` and `n4sid`) that are not taking into account the prior knowledge.

	`uy2ss_pk`	`uy2ss`	`n4sid`
$\varepsilon_{\mathscr{B}}$	0.1385	0.1572	0.1031
ε_h	0.1358	0.1565	—
$\varepsilon'_{\mathscr{B}}$	0.0025	0.0069	0.0034
ε'_h	0.0000	0.0068	—

Table 1: Average relative estimation errors $\varepsilon_{\mathscr{B}}$ and ε_h and absolute residual errors $\varepsilon'_{\mathscr{B}}$ and ε'_h for the subspace method using one equality constraint as a prior knowledge `uy2ss_pk`, not using the prior knowledge `uy2ss`, and for the N4SID method `n4sid`.

5.3 Multiple equality constraints

The results of the Monte-Carlo simulation in Section 5.2 show that prior knowledge in the form of a single equality constraint improves the estimation accuracy of both the impulse response as well as the identified system. In this section, we show the estimation errors as a function of the number of equality constraints. We use the simulation setup described in Section 5.1 with the matrix E now chosen as (5). In this case, $F = E\bar{H}$ is the vector of the first i samples of the step response of the true data generating system $\bar{\mathscr{B}}$.

The results in Figure 1 show that the estimation error for the subspace method using the prior knowledge reduces to zero as the number of equality constraints become $i = 10$. Indeed, in the case $i \geq \eta + 1$, $\bar{H}$ can be computed by solving the system of linear equations $EH = F$, without using the (noisy) data w_{d}. In general, the approximation error of Algorithm 3 (`uy2ss_pk`) decreases when more constraints are used, while the approximation error of the alternative methods that are not using the prior information is independent of the number of constraints.

Figure 2 shows the satisfaction of the equality constraint $EH = F$ in terms of the absolute residual errors ε'_h and $\varepsilon'_{\mathscr{B}}$. As in the case of a single equality constraint, the results show that the equality constraints are satisfied exactly on the first step of Algorithm 3 (`uy2ss_pk`) and although they are only approximately satisfied by the estimated model of Algorithm 3 (see Note 4), the residual error is smaller than that of the alternative methods.

5.4 Inequality constraints

In this section, we consider prior knowledge in the form of upper and lower bounds on the impulse response. The simulation setup is as described in Section 5.1, however, now $\eta = 20$ samples of the impulse response are estimated

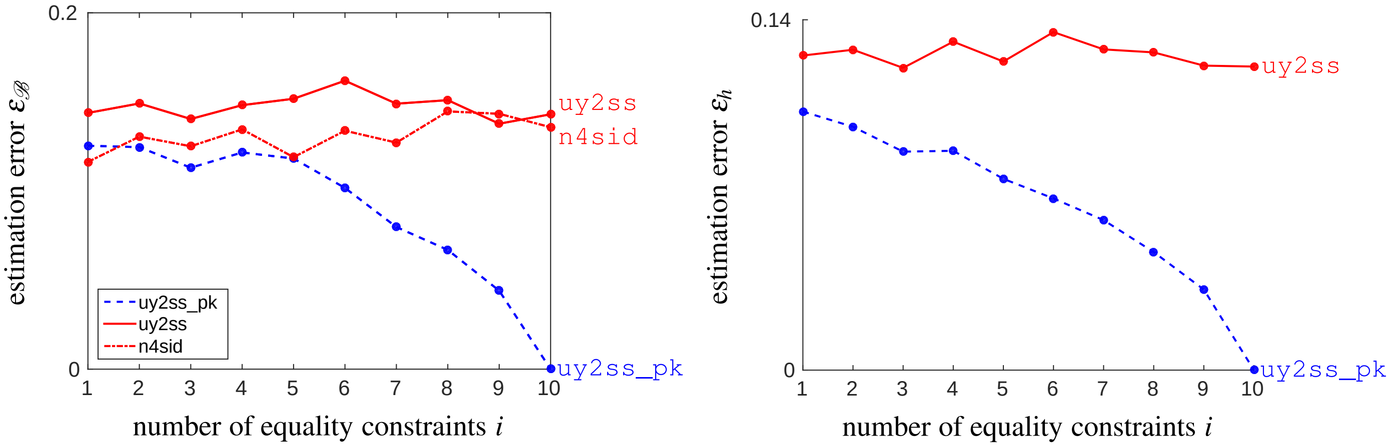

Figure 1: The approximation error of Algorithm 3 (`uy2ss_pk`) decreases when more constraints are used. In contrast, the approximation error of the alternative methods that are not using the prior information is independent of the number of constraints.

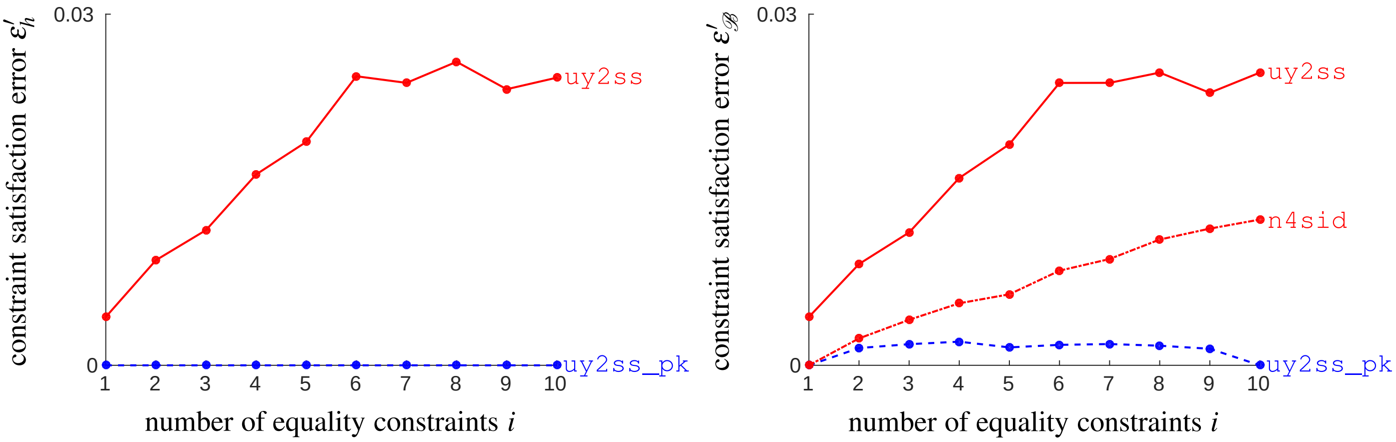

Figure 2: Left: the equality constraints are satisfied exactly on the first step of Algorithm 3 (`uy2ss_pk`). Right: The equality constraints are better satisfied by the estimated model of Algorithm 3 then they are by the alternative methods (`uy2ss` and `n4sid`).

and the trajectory w_d has $T = 100$ samples. The prior knowledge about the impulse response and the true impulse response $\bar{h}$ are shown in Figure 3, left. On the same plot are superimposed the estimated impulse responses by Algorithms 2 (`uy2h`) and 3 (`uy2h_pk`). Figure 3, right shows the true impulse response and the impulse responses of the models identified by the three methods compared: `uy2ss_pk`, `uy2ss`, and `n4sid`. The numerical values of the average relative estimation errors $\varepsilon_{\mathscr{B}}$ and ε_h are given in Table 2. The results empirically confirm the advantage of using the given prior knowledge on the impulse response for the overall identification problem.

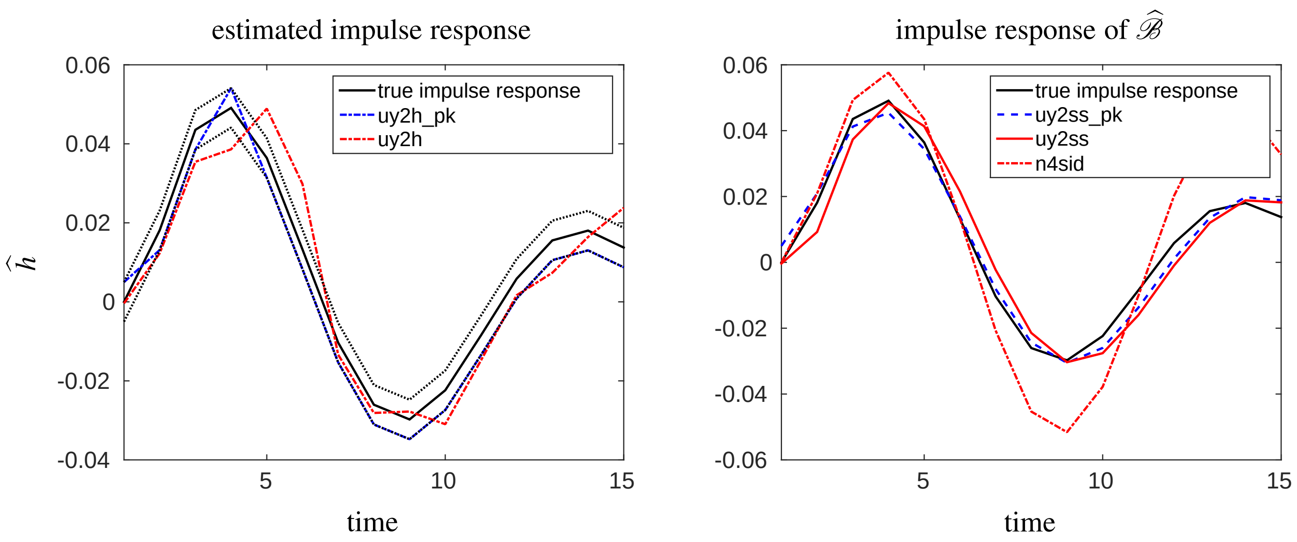

Figure 3: Estimation with inequality constraints. Left — true impulse response (solid line), prior knowledge (dotted lines), and estimates by Algorithm 3 with (`uy2h_pk`) and without (`uy2h`) prior knowledge; Right — estimated impulse responses. The constraints are satisfied by the model identified with Algorithm 3 but they are violated by the model identified with the N4SID method.

	`uy2ss_pk`	`uy2ss`	`n4sid`
$\varepsilon_{\mathscr{B}}$	0.2291	0.2611	1.4984
ε_h	0.2022	0.3235	—

Table 2: Average relative estimation errors $\varepsilon_{\mathscr{B}}$ and ε_h for the methods using inequality constraints as a prior knowledge (`uy2ss_pk`), and not using the prior knowledge (`uy2ss` and `n4sid`).

6 Conclusions

We considered prior knowledge about the to-be-identified model in the form of linear equality and inequality constraints on the impulse response. The method proposed has two steps: 1) estimation of the impulse response, and 2) realization of the estimated impulse response. Using the data-driven method for impulse response estimation of [11], incorporating prior knowledge involves solution of a convex quadratic programming problem. In case of equality constraints only, the problem allows an analytic solution. In the more general case of equality and inequality constraints, the problem can be solved globally and efficiently by existing optimization methods. The resulting algorithm has computational complexity that is comparable to that of classical subspace algorithms. Numerical examples show the improved estimation accuracy as a result of using the prior knowledge. Statistical analysis of the proposed computational method (consistency, uncertainty bound, *etc.*) is a topic of future work.

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