Improved FRF measurements of lightly damped systems using local rational models

Dries Peumans, Student Member, IEEE, Cedric Busschots, Student Member, IEEE, Gerd Vandersteen Senior Member, IEEE, and Rik Pintelon Fellow, IEEE

Abstract—Lightly damped systems exhibit strong resonances which should be accurately characterised to prevent potential harmful damage in real-world applications. Characterising these resonances using frequency response function measurements is challenging due to long transient behaviour and spectral leakage. Local modelling techniques exist which allow to remedy these difficulties but they introduce a bias or do not use an appropriate model structure. In this paper, we solve these problems by developing two local rational modelling techniques which remove the bias on the frequency response function measurement. The proposed techniques involve the use of the bootstrapped total least squares estimator on the one hand and the incorporation of prior knowledge of the pole locations on the other hand. Furthermore, the performance of both techniques is demonstrated by measuring the flexural vibrations of a steel beam.

Index Terms—Frequency response function, lightly damped system, local rational models, bootstrapped total least squares, vector fitting

I. INTRODUCTION

Frequency Response Function (FRF) measurements provide valuable insights into the dynamic behaviour of systems. In various engineering applications, these measurements are essential to understand, predict and control physical processes [1], [2], [3]. Accurate modelling of these processes is important since it enables to efficiently design, prototype and develop (new) products.

Acquiring a precise non-parametric FRF for lightly-damped systems from measured input-output data is challenging. The principal adversaries which distort the quality of FRF measurements under arbitrary excitations are transient phenomena, disturbing noise and leakage [4]. Luckily, their effects can be partly mitigated by averaging, windowing, using periodic excitations, increasing the measurement time ... [5]. However, some of these solutions are not always directly applicable in daily practice due to constraints imposed by the measurement setup and maximum available measurement time.

More advanced local modelling techniques such as the Local Polynomial Method (LPM) and the Local Rational Method (LRM) were introduced to reduce the impact of the aforementioned problems [6], [7]. In essence, both techniques approximate in a local frequency band the FRF and/or transient term with a low degree polynomial (LPM) or rational function (LRM). The accuracy of these proposed approximations rely on the observation that the FRF and transient term exhibit smooth behaviour as a function of frequency.

Although both techniques showed promising results in the past, they have several disadvantages which retain them from being generally applicable. For instance, the LPM inadequately captures strong resonance phenomena present in lightly damped systems even though an asymptotically unbiased estimator is used during the local modelling [8]. The LRM on the other hand discards the unbiasedness of the estimator by using a rational function to improve the modelling of these resonances [7]. However, this unbiasedness is crucial to correctly estimate the noise variance. Accurate knowledge of this noise variance is essential for generating confidence bounds on the FRF measurement and identifying a parametric model which uses the variance as non-parametric weighting in the cost function.

Our objective was to develop two similar local modelling techniques which allow the use of rational functions without introducing a bias on the estimate. In the first technique, we retrieved this unbiased local rational model by using the iterative Bootstrapped Total Least Squares (BTLS) estimator [9], [10]. Compared with the Least Squares (LS) solution obtained with the LRM, the BTLS estimator achieves nearly Maximum Likelihood (ML) properties by iteratively updating its estimate. Unfortunately, application of the BTLS estimator in regions without resonances occasionally results in pole-zero cancellations which greatly deteriorate the performance of the estimate. Therefore, we used a model order selection procedure which limits the presence of these cancellations as much as possible in the local model. The second technique incorporates prior knowledge of the pole locations in the traditionally used LS estimation [11]. By doing so, pole-zero cancellations can be avoided altogether and use of the more time-consuming BTLS method is unnecessary. Nevertheless, the prior knowledge of the pole locations is a knife that cuts both ways: depending on the accurateness of the information at hand it can improve or deteriorate the local model. To cope with this issue, we also developed an approach based on Vector Fitting (VF) which allows to non-parametrically extract this information in a reliable way [12].

This paper is organised as follows: Section II discusses the traditional LPM and LRM, and showcases their performance on a simulation example of a lightly damped system. Section III introduces the BTLS estimator, explains how this estimator can be efficiently implemented and proposes a model order selection procedure which limits pole-zero cancellations in the local model. Section IV demonstrates how the VF technique can be extended such that the pole locations can be non-
Fig. 1: Starting from measured input-output data (top figures), a local model \( G(\Omega_k + \delta) \) is estimated in a certain frequency window \( \Delta \Omega \) around the center frequency \( \Omega_k \). The required estimate \( G(\Omega_k) \) is obtained by evaluating \( G(\Omega_k + \delta) \) at \( \delta = 0 \).

parametrically determined starting from measured input-output data. In Section V, we apply and compare the performance of the two introduced techniques on a measurement example of a lightly damped system. Finally, we present conclusions in Section VI.

II. TRADITIONAL LOCAL MODELLING METHODS

The key idea of local modelling methods is to approximate the FRF \( G(\Omega) \) and/or the transient term \( T(\Omega) \) with a local model over the generalised frequency \( \Omega \) [4]. This generalised frequency is completely determined by the underlying processes and equals \( j\omega \) for continuous-time systems, \( \exp(j\omega T_s) \) for discrete-time systems and \( \sqrt{j\omega} \) for diffusion phenomena. The local model can be any continuous function, but has been mainly restricted to polynomials (LPM) and rational functions (LRM). In this Section, we explain the underlying idea of these local modelling methods (A), reveal how noisy input data can be managed (B) and demonstrate the performance of the LPM and LRM on a simulation example (C).

A. The basic idea

Almost all these modelling techniques start from the fundamental equation which relates the input-output discrete Fourier transform spectra \( U(k) \) and \( Y(k) \) in the frequency domain [4]

\[
Y(k) = G(\Omega_k) U(k) + T(\Omega_k) + V(k)
\]

(1)

where \( V(k) \) is an independent (over \( k \)), circular complex distributed noise source (\( \mathbb{E}\{V^2(k)\} = 0 \)) which is characterised by its noise variance \( \sigma^2_v(k) = \mathbb{E}\{ |V(k)|^2 \} \) (\( \mathbb{E}\{\cdot\} \) is the expected value operator). We define the discrete Fourier transform spectrum \( X(k) \) of \( N \) samples of an arbitrary time domain signal \( x(t) \) as

\[
X(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(tT_s) e^{-j\Omega_k tT_s}
\]

(2)

where \( T_s \) is the sampling time and \( \Omega_k = 2\pi k/(NT_s) \). Distinguishing the terms \( G(\Omega_k) U(k) \) from \( T(\Omega_k) \) demands that the input \( U(k) \) varies unpredictably over the frequency due to the smoothness of both \( G \) and \( T \). Imposing this condition on the input signal requires that the spectral difference

\[
|U(k) - U(k-1)| = O(N^0)
\]

(3)

does not vanish to zero for \( N \to \infty \). Signals which satisfy this 'roughness' condition are among others periodic noise, random noise, random-phase multisine excitations and (pseudo-)random binary sequences [4].

Since \( G(\Omega_k) \) and \( T(\Omega_k) \) are presumed to be smooth functions of \( \Omega_k \), they can be approximated locally by, for example, a rational function in the band \( r \in [-n\Delta\Omega, n\Delta\Omega) \). The local frequency variable around \( \Omega_k \) is estimated in a certain frequency window \( \Delta \Omega \)

\[
G(\Omega_{k+\delta}) = G(\Omega_k + \delta) \approx \frac{\sum_{n=0}^{N_k} b_{k,n} \delta^n}{\sum_{n=0}^{N_k} a_{k,n} \delta^n} = \frac{B_k(\delta)}{A_k(\delta)}
\]

(4)

\[
T(\Omega_{k+\delta}) = T(\Omega_k + \delta) \approx \frac{\sum_{n=0}^{N_k} c_{k,n} \delta^n}{\sum_{n=0}^{N_k} a_{k,n} \delta^n} = \frac{C_k(\delta)}{A_k(\delta)}
\]

(5)

where \( \delta \) represents the local frequency variable around \( \Omega_k \), \( 2n\Delta\Omega + 1 \) is the width of the local frequency window \( \Delta \Omega \) and \( a_{k,n}, \ b_{k,n}, \ c_{k,n} \) are complex-valued coefficients. The model equations incorporate the shared pole locations of both \( G \) and \( T \) by using the same denominator polynomial \( A_k(\delta) \) [4].

Closer inspection of (4) shows that the proposed rational function is nonlinear in the parameters which unfortunately does not allow the straightforward application of the linear least squares procedure. However, Levy (1959) proposed to weigh (1) with the denominator polynomial \( A_k(\delta) \) such that this procedure can still be used [13]

\[
A_k(\delta) Y(k+r) = B_k(\delta) U(k+r) + C_k(\delta) + A_k(\delta) V(k+r)
\]

(5)

Introduction of this weighting comes with some important disadvantages: \( A_k(\delta) \) shapes the noise term \( V(k+r) \) such that high-frequency noise is amplified and the least squares procedure results in a biased estimate of the parameters. This introduced bias becomes increasingly dominant for low signal-to-noise ratios. In these situations, an unbiased estimator would be extremely valuable since it would provide a better non-parametric estimate of the FRF and the disturbing noise variance.

The coefficients \( a_{k,n}, \ b_{k,n}, \ c_{k,n} \) can be estimated from the measured input-output data via a linear least squares procedure by equating (5) for \( r = -n\Delta\Omega, ..., n\Delta\Omega \) such that an overdetermined set of \( 2n\Delta\Omega + 1 \) local equations is obtained

\[
J\theta = \begin{pmatrix} J_{\bar{a}} & J_{\bar{b}} & J_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} \approx 0
\]

(6)
which could negatively impact the performance of the proposed way where \( \ast \) denotes the element-wise multiplication operator and \( \vec{a}, \vec{b}, \vec{c} \) represent the column vectors of the corresponding coefficients in (4). The Jacobian matrix \( J \) can be constructed based on the measured input-output data

\[
\begin{align*}
J_{\vec{a}} &= (\vec{Y} \odot \vec{\Delta} \odot \vec{\Delta} \ldots \odot \vec{\Delta}^{N_a} \odot \vec{\Delta}^{N_b}) \\
J_{\vec{b}} &= -(\vec{U} \odot \vec{\Delta} \odot \vec{\Delta} \ldots \odot \vec{\Delta}^{N_b} \odot \vec{\Delta}^{N_c}) \\
J_{\vec{c}} &= -(1 \odot \vec{\Delta} \ldots \odot \vec{\Delta}^{N_c})
\end{align*}
\]

where \( \odot \) is the element-wise multiplication operator and \( \vec{\Delta}, \vec{U}, \vec{Y} \) are the stacked column vectors of respectively the local frequencies, input and output which are present in the local frequency window \( \Delta \Omega \). Using the singular value decomposition of \( J \), the solution of (6) can be derived in a numerically stable way [4]. Furthermore, the estimation of the noise variance requires an additional white noise assumption in the local frequency window. For this assumption to be valid, \( n_{\Delta \Omega} \) should be chosen as small as possible.

**B. Handling noisy input data**

The fundamental underlying assumption made by all these local modelling approaches is that the input observation is not perturbed with noise. Otherwise a biased estimator is obtained which could negatively impact the performance of the proposed modelling techniques. To circumvent this bias, the use of the known reference signal present in the arbitrary waveform generator is proposed in [6] to retrieve the FRF. By doing so, (1) is replaced by

\[
\begin{bmatrix}
Y(k) \\
U(k)
\end{bmatrix} = \begin{bmatrix} G_{RY} \\ G_{RU} \end{bmatrix} R(k) + \begin{bmatrix} T_{RY} \\ T_{RU} \end{bmatrix} + \begin{bmatrix} V_Y \\ V_U \end{bmatrix}
\]

(8)

where \( R(k) \) is the known reference signal, \( G_{RX} \) and \( T_{RX} \) are respectively the FRF and transient contribution from the reference \( R \) to the signal \( X \), \( V_U \) and \( V_Y \) are circular complex distributed noise sources which distort the measurements. Combining \( G_{RU} \) and \( G_{RY} \)

\[
G(\Omega_k) = G_{RY}(\Omega_k) G_{RU}^{-1}(\Omega_k)
\]

(9)

allows to obtain a consistent FRF estimate of the system.

**C. Simulation example**

The goal of the simulation example is to illustrate the (in)effectiveness of both the LPM and LRM in correctly estimating \( G(\Omega_k) \) and the noise variance \( \sigma^2(k) \). For this purpose, we constructed a continuous-time (\( \Omega = j \omega \)) open-loop simulation configuration (Fig. 2) which possesses the dynamic behaviour of the lightly damped system used in Fig. 1. Furthermore, the white noise source \( E(k) \) is filtered by a second order Butterworth bandpass filter \( H(\Omega) \) before applying it to the noise-free output originating from \( G(\Omega) \)

\[
\begin{align*}
G(\Omega) &= 0.1 (\Omega^2 + 0.1 \Omega + 625) \\
H(\Omega) &= \frac{39.5 \Omega^2}{(\Omega^2 + 3.8 \Omega + 177.1)(\Omega^2 + 5.1 \Omega + 316.8)}
\end{align*}
\]

(10)

A Random-Phase Multisine (RPM) excitation signal allows to eliminate the system transient contribution from \( T(\Omega_k) \) in (1) such that we can focus entirely on the modelling capacity of both methods [4]. It consists of several simultaneously generated sinusoidal tones, described mathematically by

\[
u(t) = \sum_{\tau=1}^{F} A_{r} \sin(2\pi f_0 \tau t + \varphi_r)
\]

(11)

where \( A_r \) and \( \varphi_r \) are respectively the amplitude and phase of the \( r \)th spectral line, \( f_0 \) represents the base frequency and \( \varphi_r \) is distributed uniformly in the interval [0, 2\pi]. \( F \) is the number of tones present in the RPM. Compared with random excitations, RPMs can emulate any user-defined amplitude spectrum with \( A_r \), while conserving properties like periodicity of the signal and Gaussianity of the probability density function. These properties make them well suited for efficient simulation of systems by removing any influence of leakage and transient (\( T = 0 \)) which are related with random signals.

To retrieve an estimate of the properties \( G(\Omega_k) \) and \( \sigma^2(k) \), we applied as the input \( U(k) \) a flat RPM (all \( A_r \) are equal) with a Root Mean Square (RMS) value of 1. The signal \( E(k) \) originated from a complex normal distributed noise source. Two different experiments with a high and low Signal-to-Noise Ratio (SNR) were performed which applied respectively an RMS value of 0.0001 and 0.01. For each experiment, we averaged the results originating from respectively 50 and 10,000 Monte-Carlo simulations to deduce the general behaviour of both methods. The experiments were obtained with \( n_{\Delta \Omega} = 5 \) and we applied both the LPM and LRM with a respective model order \( N_b = 4, N_a = 0 \) and \( N_b = 2, N_a = 2 \) in (4). A parameter constraint \( a_0 = 1 \) is set to remove the parameter redundancy present in the rational representation. By doing so, the linear least squares estimation procedure results in the same number of parameters for both methods.

The modelling capacity of the LPM and LRM with a high SNR is best verified by comparing the estimated \( \sigma^2(\Omega_k) \) with the theoretically applied noise variance \( |H(\Omega_k)|^2 \sigma^2(k) \) (Fig. 3a). Visually analysing the bias on the estimated \( G(\Omega_k) \) proved to be difficult, especially in the case of the LRM where a good fit is obtained. From the simulation example we deduce that the LPM is characterised by a substantial approximation error in the vicinity of the strong resonance at 1.6 Hz. This discrepancy is caused by the polynomial model which insufficiently captures the strongly resonant behaviour. Remark that even though the LPM has a large approximation error, the mean-squared error is still a consistent measure for the reliability of the estimate.
the influence of the bias on the FRF estimate (Fig. 3b) [14]. The error $|\hat{G}_{LRM} - G|$ reveals unusual behaviour at the lower edge of the frequency grid even though identical input-output data is used during the estimation. This behaviour is a direct consequence of the changing asymmetric frequency window which alters the frequency-dependent weighting of the LS estimator [4]. Furthermore, the bias-induced effects are also visible next to the resonance frequency where $|\hat{G}_{LRM} - G|$ peaks in contrast to the standard deviation $\sigma_{LRM}$ of the mean value. In the remainder of this paper two methods will be presented that cope with the disadvantages of both the LPM and LRM.

III. Bootstrap Total Least-Squares

To overcome the disadvantages of the linear least-squares approximation, we retrieved the local rational model with an adaptation of the Weighted Generalised Total Least-Squares (WGTLS) method [15]. This method is a natural generalisation of the linear least-squares approximation and provides a consistent estimate despite noisy input-output data. The WGTLS method starts from the same linearised model equation (5) and consequently results in an identical set of equations described by the Jacobian matrix (7). It derives a solution for the estimation problem $J\theta = 0$ by solving the following optimisation problem

$$\text{argmin}_{\hat{J},\theta} \|W(J - \hat{J})C^{-1}\|_F^2$$

subject to $\hat{J}\theta = 0$ and $\theta^T\theta = 1$

where $\| \cdot \|_F$ is the Frobenius norm, $W$ is a left weighting matrix and $C$ is a square root of the column covariance matrix of $WJ$ [15]. The solution $\theta$ can be calculated through the generalised singular value decomposition of the matrix pair $(WA, C)$ instead of minimising the cost function (12).

The BTLS estimator is essentially a WGTLS estimator whose right weighting matrix $C$ is constructed to ensure consistency, while the frequency-dependent left weighting matrix $W$ is adapted iteratively to increase the efficiency [10]. By doing so, an estimator is constructed which approaches the properties of the ML estimator. However, the construction of $C$ requires knowledge of the noise variance which is not a priori available. Moreover, similar to LPM and LRM we assume that the disturbing noise in the small local frequency band is white and, hence, the column covariance matrix is known within a scaling factor $\sigma^2$. The exact knowledge of $\sigma^2$ is not required since this factor will only scale the cost function and thus does not have any effect on the retrieved solution.

Straightforward application of the BTLS estimator on the simulation example ($N_a = 2$, $N_b = 2$ and $n_{\Delta t} = 5$) results in a highly variable estimate of the noise variance (Fig. 4, middle plot). As it turns out, this unwanted behaviour is caused by pole-zero cancellations which are present in the local model $G(\Omega_b)$. To remedy these cancellations, we propose to apply a model order selection procedure in the local model. By choosing the model order as low as possible, the occurrence of pole-zero cancellations can be reduced to a minimum.

For the model order selection, we used the Akaike Information Criterion (AIC) to deduce the correct model order [16].

---

(a) Analysing the estimated FRFs $\hat{G}$ (left: LPM, right: LRM) with high SNR would suggest that only the LPM has difficulties in correctly modelling the resonance region since the error (x) of the LRM does not reveal any significant outliers compared to the standard deviation (·) of the mean FRF. Nevertheless, investigating the estimated noise variance $\hat{\sigma}_E^2$ does show a deviation from the true noise variance $|H(\Omega_k)|^2 \sigma_E^2$.

(b) Further decreasing the SNR does reveal the introduced bias by the LRM on the level of the FRF estimate $G$ (right top plot). In our simulation example, the estimate is biased in the lower frequency edge and next to the resonance frequency since the error (●) in these regions is significantly larger than the standard deviation (·) of the mean value.

Fig. 3: Applying the LPM and LRM on the simulation example with a high and low SNR allows to assess the performance of both methods. The shaded areas represent the 68% confidence interval of the estimated noise variance $\hat{\sigma}_E^2$ obtained by averaging different Monte-Carlo realisations.

due to the unbiasedness of the method [4]. The LRM on the other hand resolves this modelling issue by improving the approximating function but visibly introduces a bias in the estimated noise variance.

Lowering the SNR of the experiment significantly increases

---
The AIC decides among a group of candidate models which one describes best the local system dynamics but at the same time remains parsimonious in the number of parameters. This group of candidate models \( S \) contains all possible combinations of model orders limited to \( N_a, N_b \) and \( N_c \)

\[
S = \{ (\alpha, \beta, \gamma) \mid \alpha : 0 \to N_a, \beta : 0 \to N_b, \gamma : 0 \to N_c \}
\]  
(13)

For each of the possible model orders belonging to \( S \), the corresponding local model is derived using the BTLS estimator. From all these generated models, we retain the one which minimises the AIC cost function [17]

\[
V_{\text{AIC}}(\alpha, \beta, \gamma) = V_{\text{BTLS}} \left( 1 + \frac{n_\theta}{2n_{\Delta \Omega} + 1} \right)
\]

with

\[
V_{\text{BTLS}} = \sum_{r=-n_{\Delta \Omega}}^{n_{\Delta \Omega}} |Y(k + r) - \hat{Y}(k + r)|^2
\]  
(14)

where \( \hat{Y}(k) \) is the predicted output generated with the estimated local model, \( n_\theta = \alpha + \beta + \gamma + 2 \) is the number of unknown complex parameters and \( n_{\Delta \Omega} \) is the number of frequencies used for performing the BTLS estimation \( (2n_{\Delta \Omega} + 1 > n_\theta) \).

Combining BTLS with the AIC selection procedure significantly reduces the variability of the noise variance estimate (Fig. 4, bottom plot). Furthermore, the retrieved estimate is no longer biased as in the case of the LPM and LRM. Remark that the inclusion of the model order selection only slightly influences the error levels obtained on the FRF estimate even though the noise variance estimate is considerably improved.

The main disadvantage of the BTLS estimator is a small increase in computational cost. The computation time for estimating one local model increased from 0.59 ms with the LRM to 0.93 ms with the BTLS estimator on an Intel i7-4790 CPU (3.6 GHz). The AIC selection procedure further increases this number depending on how many candidate models are considered. In our example, 9 different candidate models were considered which resulted in an average computation time of 6.6 ms for retrieving a single estimate. This computation time is lower than the expected 9 x 0.93 ms = 8.4 ms since part of the computation can be shared over all the candidate models. For instance, the construction of the Jacobian matrix should only be done for the most complex model. All the other Jacobian matrices can be derived by simply eliminating specific columns from (7).

### IV. Incorporate Prior Knowledge

The introduction of the BTLS estimator (Section III) significantly improved the estimation of the FRF and the noise variance but unfortunately required model order selection resulting in an increased computational cost. This cost can be reduced by incorporating prior knowledge of the pole locations in the local model estimation. We propose to locally approximate \( G(\Omega_k) \) and \( T(\Omega_k) \) with the following model

\[
G(\Omega_k + \delta) \approx \sum_{n=0}^{N_b} b_n \delta^n + \sum_{n=1}^{N_p} \tilde{b}_n \delta + p_n
\]

\[
T(\Omega_k + \delta) \approx \sum_{n=0}^{N_b} \tilde{c}_n \delta^n + \sum_{n=1}^{N_p} \tilde{c}_n \delta + p_n
\]  
(15)

where \( \tilde{b}_n, \tilde{c}_n \) are the complex residues belonging to the complex pole \( p_n \). Instead of considering \( p_n \) as an unknown variable in the estimation, we set the pole location to a fixed value such that the model becomes entirely linear in the parameters. However, this decision requires that the set of poles \( \{p_n\} \) is accurately known in advance since otherwise an approximation error will be unavoidably introduced.

In general, this pole information is not available prior to the FRF measurement unless profound knowledge about the system is available through simulations or earlier characterisations. In order to reliably extract the poles \( \{p_n\} \) without any prior knowledge, we based ourselves on the iterative pole relocation algorithm used within Vector Fitting (VF) [12]. VF is a popular system identification tool which reformulates the rational approximation of the FRF as a linear least squares problem and iteratively improves this approximation by pole relocation.
In short, VF attempts to approximate the FRF $f(s)$ with a rational model [12]

$$f(s) \approx \sum_{n=1}^{N_p} \frac{r_n}{s - p_n} + d + e \, s$$  \hspace{1cm} (16)$$

where $s$ is the Laplace variable and $d$, $e$ are optional complex coefficients. Since no prior information is available about $\{p_n\}$, VF first derives these pole locations by solving the following linear problem

$$\sigma(s) \, f(s) = \rho(s)$$  \hspace{1cm} (17)$$

where

$$\sigma(s) = \sum_{n=1}^{N_p} \frac{t_n}{s - q_n} + 1$$

$$\rho(s) = \sum_{n=1}^{N_p} \frac{r_n}{s - q_n} + d + e \, s$$  \hspace{1cm} (18)$$

For a fixed set of initial poles $\{q_n\}$, (18) is linear in its unknowns $r_n$, $t_n$, $d$ and $e$. Equating (18) at different frequencies allows to generate an overdetermined set of equations which can then be solved with a linear least squares estimator. As it turns out, the poles of the resulting rational approximation are exactly equal to the zeros of $\sigma(s)$ [18] and can be efficiently derived by calculating the eigenvalues of the following matrix

$$\{p_n\} = \text{eig}(Q - I \, R^T)$$  \hspace{1cm} (19)$$

where $Q$ is a diagonal matrix containing the initial poles $\{q_n\}$, $I$ is a column vector of ones and $R^T$ is a row vector containing the residues $\{t_n\}$. This pole estimate can be further improved by iteratively substituting $\{q_n\}$ with the retrieved $\{p_n\}$.

The described VF technique cannot be straightforwardly applied to our problem setting (Fig. 2) because we do not have direct access to the FRF. Instead we adapted this technique such that $\{p_k\}$ could be locally derived from measured input-output data. Starting from (1) and substituting (16) for both $G(\Omega_k)$ and $T(\Omega_k)$ results in the following model equation

$$Y(k + r) = \left( \sum_{n=1}^{N_p} \frac{r_n}{\Omega_k + \delta - p_n} + d + e (\Omega_k + \delta) \right) U(k + r)$$

$$+ \sum_{n=1}^{N_p} \frac{r_n}{\Omega_k + \delta - p_n} + \bar{d} + \bar{e} (\Omega_k + \delta)$$  \hspace{1cm} (20)$$

where all coefficients are again complex variables. To derive the pole locations $\{p_k\}$, a similar linearised problem as in (17) is solved

$$\sigma(\Omega_k + \delta) \, Y(k + r) = \rho(\Omega_k + \delta)$$  \hspace{1cm} (21)$$

where

$$\sigma(\Omega_k + \delta) = \sum_{n=1}^{N_p} \frac{t_n}{\Omega_k + \delta - q_n} + 1$$

$$\rho(\Omega_k + \delta) = \left( \sum_{n=1}^{N_p} \frac{r_n}{\Omega_k + \delta - p_n} + d + e (\Omega_k + \delta) \right) U(k + r)$$

$$+ \sum_{n=1}^{N_p} \frac{r_n}{\Omega_k + \delta - p_n} + \bar{d} + \bar{e} (\Omega_k + \delta)$$  \hspace{1cm} (22)$$

Substituting (22) into (21) and using the frequencies present in $\Delta \Omega$ around $\Omega_k$ results in an overdetermined set of equations which can be solved via LS estimation. Afterwards, (19) can be used to retrieve an estimate of $\{p_k\}$. In a final stage the iteratively obtained pole locations are substituted in (15) and a last LS optimisation is performed to derive $b_n$, $\tilde{b}_n$, $c_n$ and $\tilde{c}_n$ with known poles. For the remainder of this paper we will denote our proposed extension as Extended Input-Output Vector Fitting (EIOVF).

To verify the performance of the adapted VF technique, we applied it again on the simulation example with $N_b = 3$, $N_p = 1$ and $\Delta \Omega = 6 \text{mHz}$ (Fig. 5, top plot). Comparing the true applied noise variance with the retrieved estimate shows a good correspondence within the 68% confidence bounds. Furthermore, the computational efficiency of this method is slightly better than that of the BTLS estimator since a computation time of 0.87 ms for a single local model is reached. However, a small deviation in the noise variance estimate is visible in the region where the pole influence is diminished. This behaviour is caused by the LS estimation which assigns a small value to $\tilde{b}_n$ even though the estimated pole location is situated far from the imaginary axis.

To cope with this issue, we used a similar model order selection technique based on AIC as in Section III. To limit the computational cost, only three different candidate models were taken into consideration instead of evaluating all the possible model order combinations from (15). The evaluated candidate models are: the complete model (15), a polynomial with order $N_b$ and a partial fraction expansion with order $N_p$. Applying this model order selection (Fig. 5, bottom plot) shows that the noise variance estimate is further improved in the anti-resonance region. As it turns out, the AIC prefers the purely polynomial model to the full model (15) in this region.
V. EXPERIMENTAL ILLUSTRATION

Verifying the proposed techniques onto a practical measurement example is essential to demonstrate the (dis)advantages in an experimental setting. In this Section, we describe the system under test, the data acquisition system and the performed experiments (A). Using the known reference signal, periodic excitations are used to measure the FRF and the noise variance for validation purposes (B). Furthermore, we validate the different techniques by analysing the estimated FRF and output noise variance under random binary excitations (C).

A. Measurement setup

The system under test is a steel beam \( (\rho = 7800 \text{ kg/m}^3) \) of length 61 cm, height 2.47 cm and width 4.93 mm, hung by two nylon threads for which we want to characterise the flexural vibrations (Fig. 6) \[4\]. These vibrations are induced by a mini-shaker (B&K 4810) which applies a transverse force at 10 cm from the end of the beam. The input of the mini-shaker originates from the VXI arbitrary waveform generator (HPE 1445A, \( Z_{\text{out}} = 50 \Omega \)) and is first amplified before connecting it resistively \((R = 18\Omega)\). This resistor is inserted to avoid the adverse influence of the inductive impedance of the mini-shaker onto the functioning of the amplifier. Both the applied input force and resulting acceleration are measured using an impedance head (B&K 8001). These signals are amplified (B&K 2635) and buffered \((Z_{\text{in}} > 5 \text{ M}\Omega, Z_{\text{out}} = 50 \Omega)\) before applying them to the VXI data acquisition system (HPE 1430A). Each slot of the VXI measurement system is internally synchronised with a sampling frequency \(f_s = 10 \text{ MHz}/2^n \approx 19.53 \text{ kHz}\).

Two different kinds of excitation signals are applied to the steel beam with an RMS value of 241 mV in a frequency band from 0Hz to 6kHz:

- a random phase multisine \((11)\) where 50 periods of \(N = 1024\) samples are averaged to generate a steady output validation set for the system which is not perturbed with a transient contribution,
- a random binary sequence which is divided into 50 blocks of 1024 samples to verify the proposed techniques under arbitrary excitations.

For each excitation signal an experiment is performed where the reference signal (signal stored in the arbitrary waveform generator), the input (force) and the output (acceleration) are collected.

B. Measurement of the steady-state validation set

Using the random-phase multisine experiment and \((8)\), we can derive a validation estimate of the FRF and the noise variances (Fig. 7). This estimation involved averaging different periods of the steady-state response \([4]\). We only retained the last 30 periods from the experiment to ensure that transient phenomena were completely eliminated from the response. Due to the mechanical nature of the system, the FRF exhibits multiple structural (anti-)resonances. This resonant behaviour is also clearly visible in the output noise variance estimate. Furthermore, we observe that a limited frequency resolution is available which will challenge the traditional local modelling approaches. For example, the LPM at least requires 7 frequency points in the 3dB bandwidth of the resonance to ensure that the approximation error remains sufficiently low \([19]\).

To verify the obtained steady-state validation set, we applied the LPM \((N_b = 4, N_c = 4 \text{ and } n_{\Delta \Omega} = 6)\) onto the full random binary sequence of \(50 \times 1024\) samples. Due to the increase of the frequency resolution with a factor 50, the LPM results in a reliable estimate of both the FRF and output noise variance. Analysing the resulting noise variance (Fig. 7) shows that the steady-state validation set only describes three of the larger resonances in the output noise spectrum due to the limited frequency resolution.

C. Comparison of the different techniques

Using random binary excitations allows to illustrate the performance of both the proposed BTLS estimator and the VF technique in a general framework since both \(G\) and \(T\) should be modelled simultaneously. Similar to the simulation example, we compare both the estimated FRF and output noise variance with the validation set (Fig. 8). Confidence bounds are generated by averaging the obtained estimates for each block of \(N = 1024\) samples.

The local model orders were chosen in such a way that all methods incorporated the same number of parameters in the local LS estimation (Table 1). However, we applied the AIC model order selection in the case of the LRM and BTLS to prevent pole-zero cancellations in the local model. Further investigation into the estimated pole locations with EIOVF (Fig. 9) shows that the earlier outlined pole estimation algorithm does behave as expected. In the resonance regions, the poles are located close to the imaginary axis and consequently have a significant contribution to \(G\) and \(T\). The flat regions do not require these contributions and thus the algorithm does not modify the initial pole locations. We used these initial pole locations as a model order selection criterion to decide whether to use exclusively polynomials or partial fractions in the final LS estimation \((15)\). In the region where the distinction is not conclusive \((150 \text{ Hz} – 700 \text{ Hz})\) we used the same AIC criterion as described in Section IV to derive the correct model order.

Applying the LPM on the experimental data clearly reveals a substantial approximation error in the vicinity of the resonances. All the other techniques significantly improve the estimate of the FRF and the output noise variance. Nevertheless,
analysing \( \hat{\sigma}_Y^2 \) shows a difference in performance of the different techniques. For instance, the EIOVF technique exhibits multiple resonances in \( \hat{\sigma}_Y^2 \) around 2.3 kHz and 4 kHz which are not present in the validation set. However, the experiment with the increased frequency resolution (Fig. 7, solid line) does reveal these multiple resonances which demonstrates that our method attempts to estimate these resonances despite the limited frequency resolution. Furthermore, the estimate of \( \hat{\sigma}_Y^2 \) at the upper frequency edge is only slightly improved compared to the LPM.

As it turns out, the LRM is the only technique that despite the bias predicts the resonance at 5.7 kHz, even though this resonance is not present in the validation set. Furthermore, the LRM reveals the same unusual behaviour at the edges of the frequency grid as outlined in Section II-A. All the other local modelling techniques correctly exhibit a constant behaviour in these regions since an unbiased estimator is used.

Compared with all other techniques, the BTLS estimator best approaches the validation set at the edges and furthermore correctly reveals the three resonances. Nevertheless, the estimates of the resonance regions are broader than expected which is a correct indication of the multiple resonances.

VI. Conclusions

The local modelling techniques introduced in this paper allow to accurately measure the FRF and noise variance under arbitrary excitations. Both the BTLS estimator and incorporation of prior knowledge using Vector Fitting permit to effectively estimate local rational models without introducing a bias. Nevertheless, these techniques slightly increased the computational cost due to model order selection and estimation of the pole locations with an adaptation of the VF technique. The performance of both techniques was demonstrated by experimentally characterising the flexural vibrations of a steel beam. In both cases, good correspondence with the validation set was obtained despite the very low frequency resolution.

Acknowledgment

This work is sponsored by the Fund for Scientific Research (FWO Vlaanderen), the Flemish Government (Methusalem Fund METH1), the Strategic Research Program of the VUB (SRP-19) and the Belgian Federal Government (IUAP VII).

References


Dries Peumans (1992, Belgium) graduated as an Engineer in Electronics and Information Technology in 2015 at the VUB. Afterwards he joined the department ELEC as a PhD researcher. His research focuses on nonlinear modelling and the design of analog circuits.

Cedric Busschots (1990, Belgium) obtained the degree of ‘industrieel ingenieur in de elektronica-ICT’ at Lessius Mechelen - Campus De Nayer. He continued at the VUB and ULB, and graduated in 2015 with a degree in electromechanical engineering. In August 2015, he joined the ELEC department as a PhD researcher. His research focuses on system identification using microcontrollers.

Gerd Vandersteen (1968, Belgium) received the degree in electrical engineering in 1991 and his PhD in electrical engineering in 1997, both from the VUB, Brussels, Belgium. He was with the Micro-Electronics Research Centre IMEC, Wireless Group, as a Principal Scientist, with a focus on modeling, measurement, and simulation of electronic circuits in state-of-the-art silicon technologies. This research was in the context of a collaboration with the VUB. Since 2008, he has been a Professor with the Department of Fundamental Electricity and Instrumentation, VUB, within the context of measuring, modelling and analysis of complex linear and nonlinear system. Within this context, the set of systems under consideration is extended from micro-electronic circuits towards all kinds of electro-mechanical systems. Since 2011, he has been the director of the Doctoral School of Natural Sciences and (Bioscience) Engineering (NSE), VUB.

Rik Pintelon received the masters degree in electrical engineering in 1982, a doctorate (Ph.D.) in engineering in 1988, all from the Vrije Universiteit Brussel (VUB), Brussels, Belgium. In 2014 he received the degree of Doctor of Science (D.Sc.) from the University of Warwick (UK) for publications with the collective title ‘Frequency Domain System Identification: A Mature Modeling Tool’. From 1982 to 1984 and 1986 to 2000, Dr. Pintelon was a researcher with the Belgian National Fund for Scientific Research (FWO-Vlaanderen) at the Department of Electrical Engineering (ELEC) of the VUB. From 1984 to 1986 he did his military service overseas in Tunisia at the Institut National Agronomique de Tunis. From 1991 to 2000 he was a part-time lecturer at the Department of ELEC of the VUB, and since 2000 he is a full-time professor in electrical engineering at the same department. Since 2009 he is a visiting professor at the Department of Computer Science of the Katholieke Universiteit Leuven, and since 2013 he is a honorary professor in the School of Engineering of the University of Warwick. His main research interests include system identification, signal processing and measurement techniques. Dr. Pintelon is the coauthor of 4 books on system identification and the coauthor of more than 200 articles in refereed international journals. He has been a Fellow of IEEE since 1998. Dr. Pintelon was the recipient of the 2012 IEEE Joseph F. Keithley Award in Instrumentation and Measurement (IEEE Technical Field Award). He received the 2008 IOP outstanding paper award (best paper in Measurement Science & Technology), the 2014 Martin Black prize (best paper in Physiological Measurement) and the 2014 Andy Chi award (best paper in IEEE Trans. on Instrumentation and Measurement).
Fig. 8: Comparing the FRF and noise variance estimates of the different techniques with the validation set (RPM) allows to evaluate simultaneously both the advantages and disadvantages of each technique. Again, 50 different measurements ($N = 1024$) are averaged to retrieve the 68% uncertainty bounds.

Fig. 9: In the resonance regions, the EIOVF technique correctly positions the estimated poles close to the imaginary axis.