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Accurate estimation of the non-parametric FRF of lightly-damped mechanical systems using arbitrary excitations

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Abstract

Lightly-damped mechanical systems exhibit strong resonant behaviour which could potentially result in life-threatening situations. To prevent these situations from happening, frequency response function measurements are essential to accurately characterise the resonant modes of the mechanical system. Unfortunately, these measurements are distorted by leakage and (long) transient phenomena. Local modelling techniques have been introduced in the past to resolve these complications but either they do not use the correct model structure or they introduce a bias. This paper proposes a local rational modelling technique which completely removes the bias from the estimation procedure and is applicable to large-scale multiple-input, multiple-output systems. The developed technique uses the bootstrapped total least squares estimator which provides unbiased estimates for the local rational model and generates accurate uncertainty bounds for the obtained non-parametric frequency response function. The proposed technique is successfully verified using a simulation example of a large-scale system which contains 100 resonances and has 100 inputs and 100 outputs. Its practical applicability is illustrated by characterising the resonant behaviour of the tailplane of a glider.

Keywords:
Frequency response function, lightly-damped mechanical system, local rational modelling, multiple-input multiple-output, non-parametric noise model

1. Introduction

Most mechanical structures such as buildings, bridges, airplanes... are very susceptible to resonant behaviour. Mechanical resonances can be induced by proper application of small forces which in turn result in excessive oscillatory motions of the structure [1]. Depending on the applications at hand, these oscillations are desirable (e.g. pendulum, musical instruments ...) or potentially life-threatening (e.g. collapsing bridges, flight flutter ...).

The Frequency Response Function (FRF) is a fundamental quantity which heavily facilitates the study of the dynamic behaviour of these mechanical structures [1, 2]. By correctly measuring the non-parametric FRF and its corresponding uncertainty, a parametric model can be estimated which enables the extraction of modal parameters such as resonance frequency, damping ratio ... [3, 4, 5]. Thorough knowledge of these modal parameters is crucial to deduce whether or not a mechanical structure will exhibit oscillatory motion when taken into operation. In addition, these parameters allow engineers to counteract or control this possibly destructive behaviour. Although the FRF is a well-established concept within many engineering disciplines, accurately estimating it for lightly-damped systems with arbitrary excitations remains challenging. The primary adversaries which significantly deteriorate the quality of the FRF measurement of these
systems are leakage, long-lasting transient phenomena and noise. Fortunately, these side effects can be somewhat diminished by windowing (leakage reduction), averaging (noise reduction), applying periodic excitations (leakage removal and noise reduction), increasing the measurement time ... provided that either the measurement setup or the available measurement time allow for these countermeasures [2, 6, 7].

Recently, advanced local modelling techniques, such as the Local Polynomial Method (LPM) [8, 9] and the Local Rational Method (LRM) [10, 11], have been developed which cope with the above-mentioned challenges. Starting from measured noisy input-output data, these methods locally approximate the FRF and transient term in the frequency domain by a polynomial (LPM) or rational function (LRM). By doing so, accurate estimates of the non-parametric FRF have been obtained in a myriad of applications [6, 12]. Unfortunately, both techniques have different shortcomings that prevent them from being generally applicable. For instance, the LPM is ill-suited for lightly-damped mechanical systems since the local polynomial model insufficiently captures resonant behaviour unless a disproportionate measurement time is used [13]. The LRM on the other hand significantly improves the modelling capacity of these resonances by using a local rational model. This improvement unfortunately does not come for free: the rational model is derived using a biased estimator [10, 11] and the extension towards Multiple-Input, Multiple-Output (MIMO) is not trivial due to various possible parametrisations of the FRF matrix [6, 11, 14, 15]. The ability of a method to estimate an unbiased local model is essential for generating confidence bounds on the FRF [8], while the identification of mechanical systems with arbitrary MIMO excitations ensures that all the resonances of the structure are properly excited [16].

To remedy the above-mentioned shortcomings of currently existing local modeling techniques, we develop in this paper a local rational modelling technique based on the iterative Bootstrapped Total Least Squares (BTLS) estimator which provides an unbiased estimate of the non-parametric FRF and is applicable to large-scale MIMO systems. By doing so, both the FRF and its corresponding uncertainty can be accurately estimated at the expense of an increased computational cost compared with LPM and LRM.

Summarised, the contributions of this paper are

1. the positioning of our work in relation to currently existing local modelling techniques (Section 2),
2. the adaptation of the iterative BTLS estimator [17, 18, 19] to retrieve an unbiased local rational model with uncertainty bounds (Section 3),
3. the computationally efficient calculation for large-scale MIMO systems (Section 4),
4. the comparison with state-of-the-art local modelling techniques on a large-scale MIMO system (100 inputs and 100 outputs) which contains 100 resonances (Section 5),
5. the characterisation of the resonant behaviour of the tailplane of a glider (Section 6).

2. Existing local modelling techniques for non-parametric FRF estimation

Nearly all currently existing local modelling techniques start from the same fundamental equation in the frequency domain which exactly describes the input-output behaviour of a linear MIMO system with \( n_u \) inputs and \( n_y \) outputs (Fig. 1)

\[
Y(k) = G(\Omega_k) U(k) + T(\Omega_k) + V_Y(k)
\]  

(1)

where \( U(k) \in \mathbb{C}^{n_u \times 1} \) and \( Y(k) \in \mathbb{C}^{n_y \times 1} \) are the discrete Fourier transform spectra of respectively the input and output of the system evaluated at the frequency bin \( k \).

\(^1\)All vectors and matrices are represented by boldface variables.
Figure 1: The output-error framework describes the input-output behaviour of a linear dynamic system in the frequency domain. The system is assumed to be excited with a known input $U(k)$ while the output $Y(k)$ is perturbed by the noise disturbance $V_Y(k)$.

We define the discrete Fourier transform spectrum $X(k)$ of $N$ samples of an arbitrary time domain signal $x(t)$ as

$$X(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-j\omega_k t T_s}$$

where $T_s$ is the sampling period and $\omega_k = 2\pi k/(NT_s)$. Note that $X(k)$ does not necessarily have to be derived from time-domain signals but can also be directly measured with, for example, network analysers. $G(\Omega_k) \in \mathbb{C}^{n_y \times n_u}$ and $T(\Omega_k) \in \mathbb{C}^{n_y \times 1}$ are, respectively, the multivariable FRF matrix and transient contribution as a function of the generalised frequency variable $\Omega_k$. Depending on the underlying process, $\Omega_k$ equals $j\omega_k$ for continuous-time systems, $e^{-j\omega_k T_s}$ for discrete-time systems and $\sqrt{j\omega_k}$ for diffusion phenomena.

The output is perturbed by the noise disturbance $V_Y(k)$, while the input is assumed to be exactly known (output-error framework). Furthermore, we assume that the noise source $V_Y(k)$ has the following properties:

1. $V_Y(k)$ is uncorrelated with $V_Y(r)$ whenever $k \neq r$,
2. $V_Y(k)$ is independent of the input signal $U(k)$ which automatically implies that the system under study should operate in open-loop.

Closed-loop identification of MIMO systems is possible, but requires the exact knowledge of the applied reference signals which are, for example, the signals stored into the arbitrary waveform generators [20].

The smoothness of both $G(\Omega_k)$ and $T(\Omega_k)$ requires that $U(k)$ varies unpredictably as a function of the frequency such that the local modelling technique can discriminate between $G(\Omega_k) U(k)$ and $T(\Omega_k)$. This requirement implies a ‘roughness’ condition on the input signal for which the element-wise spectral difference ($U_i$ is the $i^{th}$ element of $U$)

$$|U_i(k+1) - U_i(k)| \quad i = 1, 2 \ldots n_u$$

should not vanish to zero for $\Delta \omega_k \rightarrow 0$ where $\Delta \omega_k = \omega_{k+1} - \omega_k$. Even though the local modelling techniques are applicable to both random and deterministic (periodic and non-periodic) excitations, we focus only on signals which satisfy this ‘roughness’ condition. Such signals are among others random noise, periodic noise, random-phase multisine excitations and (pseudo-)random binary sequences. Another requirement for these signals is that they guarantee identifiability of each local model in the MIMO case. A necessary condition for this requirement is that at each frequency none of the inputs $U_i$ can be fully correlated with the other inputs $U_j$. Mathematically, this condition is satisfied when $\mathbb{E}[U(k)U(k)^H]$ is full rank [6]. Here, $\mathbb{E}[\cdot]$ is the expected value operator with respect to a random realisation of $U$ and $\cdot^H$ is the conjugate transpose operator.
The key idea of the local modelling techniques is to approximate both $G(\Omega_k)$ and $T(\Omega_k)$ with a local model over $\Omega_k$. Since $G(\Omega_k)$ and $T(\Omega_k)$ are presumed to be smooth functions of $\Omega_k$, essentially any continuous function can be used for this purpose. Popular choices which showed outstanding results in the past are polynomial matrices (LPM) and rational matrix parametrisations (LRM) [8, 9, 11]. To showcase the mechanism behind these local modelling techniques, let us pretend that we want to estimate $G(\Omega_k)$ and $T(\Omega_k)$ for a SISO system by using a polynomial model (Fig. 2). Using the local frequency band $[\Omega_k-n\Delta\Omega \ldots \Omega_k+r \ldots \Omega_k+n\Delta\Omega]$, both the FRF and transient term in (1) can be approximated by polynomials

$$G(\Omega_{k+r}) = G(\Omega_k + \delta_r) \approx \hat{G}(\Omega_k) + \sum_{l=1}^{n_G} g_{kl} \delta_r^l$$

$$T(\Omega_{k+r}) = T(\Omega_k + \delta_r) \approx \hat{T}(\Omega_k) + \sum_{l=1}^{n_T} t_{kl} \delta_r^l$$

(4)

where $\delta_r$ quantifies the local frequency variation around $\Omega_k$, $2n\Delta\Omega + 1$ is the number of frequency points considered for the local estimation within the local frequency window $\Delta\Omega$. $\hat{G}(\Omega_k)$ and $\hat{T}(\Omega_k)$ are, respectively, the local FRF and transient estimates while $g_{kl} \in \mathbb{C}^{n_y \times n_u}$ and $t_{kl} \in \mathbb{C}^{n_y \times 1}$ are the unknown polynomial coefficients which model the dynamic variations within each local model. Since these dynamic variations are limited in the local modelling context, the polynomial orders $n_G$ and $n_T$ do not exceed 4 in most use cases [6]. Furthermore, the same order is mostly used for $G$ and $T$ ($n_G = n_T$) such that identical modelling capacities can be provided for both terms. The coefficients $g_{kl}$ and $t_{kl}$, together with $\hat{G}(\Omega_k)$ and $\hat{T}(\Omega_k)$, can be retrieved by substituting (4) in (1) and consequently solving the resulting set of linear equations. By shifting the local window over the whole frequency range, it is possible to retrieve an estimate for $G$, $T$ and
the covariance matrix $\text{Cov}(Y)$ of the noise perturbation.

The identification of an unbiased local rational model for MIMO systems has two major challenges: the rational function matrix parametrisation is not unambiguously defined [14], and the estimation procedure is difficult since the model is inherently nonlinear in the parameters. Many different parametrisations of the rational function matrix exist such as the state-space representation, matrix and partial fraction descriptions [14]. Depending on the application at hand, one parametrisation is better suited to describe $G$ and $T$ than others.

The left matrix fraction (LMF) description is commonly adopted since this parametrisation leads to a linear relationship between all the matrix coefficients. The left matrix fraction description rewrites the nonlinear rational function matrices $G$ and $T$ as the ratio of two matrix polynomials

$$G(\Omega_k + \delta_r) \approx A_k(\delta_r)^{-1} B_k(\delta_r) = \left( \sum_{l=0}^{n_A} a_{kl} \delta_r^l \right)^{-1} \left( \sum_{l=0}^{n_B} b_{kl} \delta_r^l \right)$$

$$T(\Omega_k + \delta_r) \approx A_k(\delta_r)^{-1} C_k(\delta_r) = \left( \sum_{l=0}^{n_A} a_{kl} \delta_r^l \right)^{-1} \left( \sum_{l=0}^{n_C} c_{kl} \delta_r^l \right)$$

where $a_{kl} \in \mathbb{C}^{n_r \times n_r}$ are the denominator complex matrix coefficients and $b_{kl} \in \mathbb{C}^{n_r \times n_u}$ and $c_{kl} \in \mathbb{C}^{n_r \times 1}$ are the numerator complex matrix and vector coefficients of, respectively, $G$ and $T$. The same denominator polynomial $A_k$ is used for both $G$ and $T$ since both terms share common poles [6]. Again, the polynomial orders $n_A$, $n_B$ and $n_C$ are mostly chosen equal to each other (with a maximum order of 4). This choice ensures that the local rational function has no favorable modelling capacity towards the resonance or antiresonance regions. The rationale behind the complex coefficients is that the reformulation of $G$ and $T$ as a function of $\delta_r$ does not yield real spectra which are typified by a complex conjugate mirrored spectrum around $\delta_r = 0$. Substituting (5) and (6) in (1) and multiplying the result with $A_k(\delta_r)$ gives the following linearised model equations for $r = -n_{\Delta \Omega}, \ldots, -1, 0, 1, \ldots, n_{\Delta \Omega}$

$$A_k(\delta_r) Y(k + r) = B_k(\delta_r) U(k + r) + C_k(\delta_r) + A_k(\delta_r) Y(k + r)$$

which can be easily solved with a linear least squares estimation procedure. This procedure only provides a consistent estimate if $Y(k + r)$ originates from a white noise source $E_y(k + r)$ which is filtered by the denominator polynomial: $Y(k + r) = A_k(\delta_r)^{-1} E_y(k + r)$. In all other cases $A_k(\delta_r)$ wrongly shapes the noise term $Y(k + r)$ such that high-frequency noise is amplified and the linear least squares estimation procedure results in a biased estimate of the model coefficients. For low signal-to-noise ratios ($< 20 \, \text{dB}$), this bias becomes increasingly dominant and undermines the general applicability of the LMF parametrisation for local modelling purposes [21].

3. Using BTLS to retrieve an unbiased local rational model

A wide variety of estimators have been developed in the past to tackle the problem of estimating a rational transfer function matrix with the LMF parametrisation [22]. Most of these frequency domain estimators have been studied and successfully applied in a global identification setting where the objective consisted of capturing the whole dynamic behaviour of the FRF with one single parametric rational transfer function matrix. The identification of this FRF matrix is difficult for lightly-damped mechanical systems but can be made possible even when excessive orders of $A_k$, $B_k$ and $C_k$ are demanded due to a large number
of resonant modes [23]. Other identification approaches exist which do not require an initial non-parametric FRF identification but use the input/output data directly to acquire a global parametric model [24].

The main reasons why accurate non-parametric models are required in the global identification setting are the following

1. After the global identification stage, the obtained parametric model should be validated with an accurate non-parametric estimate to detect modelling errors.
2. Some of the parametric identification procedures heavily rely on the accuracy of the non-parametric noise model during their estimation [6].

In both cases, the local modelling techniques contribute by significantly improving this non-parametric estimation. Using the least squares estimator within this local modelling context seemed favorable at first sight: the estimator is fast and computationally efficient implementations exist [6]. Unfortunately, this estimator is biased and introduces an erroneous noise shaping which becomes even more significant for large-scale MIMO systems where an increasing number of frequencies are needed to identify all the local model parameters.

Iterative methods have been introduced to improve the least squares estimate with varying precision [6]. The most straightforward one is Sanathanan-Koerner [25, 26], which overcomes the lack of sensitivity towards the low-frequency errors by iteratively reweighting the least squares solution with the denominator polynomial obtained in a previous iteration. However, recent applications of this method for local modelling purposes [11, 21] have resulted in estimation errors which are larger than the non-iterative LRM due to convergence issues and over-fitting, making this method ill-suited to perform the local model estimation. More advanced gradient-based methods exist but require a Gauss-Newton or Levenberg-Marquardt optimisation scheme to solve the nonlinear estimation problem directly [27, 28]. These schemes are not recommended in the local modelling context since they require good initial estimates to avoid local minima. Accurate generation of these initial values most often requires an additional identification step which negatively impacts the complexity of the estimation procedure.

To overcome the negative aspects associated with the above-mentioned estimation techniques, we developed a technique which allows to efficiently estimate a local rational model without introducing a bias on the estimate. We retrieved this unbiased rational model by using the iterative Bootstrapped Total Least Squares (BTLS) estimator which has primarily been used in the past for global system identification purposes [17, 18]. The main advantage of this estimator is that during each iteration step the global minimum of a cost function is computed using the generalised singular value decomposition. This results in an consistent estimate at each step, where the iterations are used to improve the efficiency of the estimator by tending towards the Maximum Likelihood (ML) cost. Furthermore, the iterative BTLS estimator is self-starting, which, compared with gradient-based methods, eliminates the dependency on other estimators for initialisation purposes.

Recently, the BTLS estimator was successfully applied in the local modelling context for the non-parametric identification of SISO systems [19]. In the remainder of this paper, the name Local BTLS (LBTLS) is adopted to describe the application of the BTLS estimator within this local modelling context. The main purpose of this section is to demonstrate the necessary steps and accompanying assumptions to expand the system class towards large-scale MIMO systems. To do so, the model equations are constructed (Section 3.1) and the minimisation problem which needs to be solved is defined (Section 3.2). Afterwards, we derive expressions for the weighting matrix and the column covariance matrix which are used by the LBTLS estimator to estimate the FRF (Section 3.3). Equally important with the consistent FRF estimation, uncertainty bounds on the estimated properties can also be derived (Section 3.4). Finally, we introduce an
augmented modelling step such that the local modelling technique can be applied on noisy input, noisy output observations of systems operating in closed-loop (Section 3.5).

3.1. The model equations

Using the earlier introduced LMF description (5) and (6), a set of linear model equations is obtained as in (7). These model equations could be solved directly using a linear least squares estimation procedure. However, multivariable systems often contain common pole dynamics which makes this LMF parametrisation too general, in the sense that it does not include any coupling between the different outputs in term of pole dynamics. To mitigate this predicament, we propose to use the common denominator parametrisation which can be derived from the LMF model by imposing structure on \( a_{kl} \)

\[
a_{kl} = a_l(k) I_{n_y}
\]

where \( a_l(k) \in \mathbb{C} \) and \( I_{n_y} \in \mathbb{C}^{n_y \times n_y} \) is the identity matrix of size \( n_y \). Remark that the structure imposed in (8) is entirely equivalent to the following common-denominator transfer-matrix representation which relates every output \( Y_j \) (\( j = 1 \ldots n_y \)) of the MIMO system with the inputs \( U_i \) (\( i = 1 \ldots n_u \)) and the transient term (for clarity the index \( k \) is omitted in the polynomial models)

\[
Y_j(k + r) = \sum_{l=1}^{n_y} B_p(\delta_r, \theta) U_i(k + r) + \frac{C_j(\delta_r, \theta)}{A(\delta_r, \theta)}
\]

where \( A(\delta_r, \theta) \) is the common-denominator polynomial, \( B_p(\delta_r, \theta) \) is the numerator polynomial from input \( U_i \) to output \( Y_j \) and \( C_j(\delta_r, \theta) \) is the numerator polynomial which captures the transient contribution for output \( Y_j \). These polynomials are defined as

\[
A(\delta_r, \theta) = \sum_{l=0}^{n_u} a_l \delta_r^l \\
B_p(\delta_r, \theta) = \sum_{l=0}^{n_u} b_p^l \delta_r^l \\
C_j(\delta_r, \theta) = \sum_{l=0}^{n_c} c_j^l \delta_r^l
\]

where \( a_l, b_p^l, c_j^l \in \mathbb{C} \) are the unknown coefficients. \( \theta \in \mathbb{C}^{((n_u+1)n_y+n_c+1)n_y+n_c+1 \times 1} \) represents the stacked vector of all unknown coefficients which can be constructed by combining the coefficients in the following manner

\[
\theta = \begin{pmatrix}
\theta_{B,Y_1} \\
\theta_{C,Y_1} \\
\theta_{B,Y_2} \\
\theta_{C,Y_2} \\
\vdots \\
\theta_{B,Y_n} \\
\theta_{C,Y_n} \\
a_0 \\
a_1 \\
\vdots \\
a_{n_y}
\end{pmatrix} \in \mathbb{C}^{n_y \times 1}
\text{ with } \theta_{B,Y_j} = \text{vec}
\begin{pmatrix}
b_{01}^j & b_{02}^j & \ldots & b_{0n_y}^j \\
b_{11}^j & b_{12}^j & \ldots & b_{1n_y}^j \\
\vdots & \vdots & \ddots & \vdots \\
b_{n_u1}^j & b_{n_u2}^j & \ldots & b_{n_un_y}^j
\end{pmatrix}
\text{ and } \theta_{C,Y_j} = \begin{pmatrix}
c_{0j}^1 \\
c_{0j}^2 \\
\vdots \\
c_{0j}^{n_y}
\end{pmatrix}
\]

where vec(\( \bullet \)) is the vector operator which stacks all the columns of a matrix on top of each other and \( n_\theta \) is the total number of coefficients. Multiplying (9) with \( A(\delta_r, \theta) \) and substituting the polynomials with their
description in (11), the original set of model equations can be restructured to a matrix format, and this for each frequency $\Omega_{k+r}$ present in the local window

$$
e(\Omega_{k+r}, \theta) = J(\Omega_{k+r}) \theta = \begin{pmatrix}
U^T \otimes P^{nB} & P^{nC} & -Y_1 P^{nA} \\
U^T \otimes P^{nB} & P^{nC} & -Y_2 P^{nA} \\
\vdots & \vdots & \vdots \\
U^T \otimes P^{nB} & P^{nC} & -Y_n P^{nA}
\end{pmatrix} \theta \approx 0 \quad (12)
$$

where $\otimes$ is the Kronecker product [29] and $P^n = \begin{pmatrix} 1 & \delta_r & \delta_r^2 & \ldots & \delta_r^n \end{pmatrix}$. Adopting the proposed common denominator structure not only results in a parametrisation with less parameters than the general LMF description (5)-(6), it also grants us the possibility to introduce structure in $e(\Omega_{k+r}, \theta)$ which facilitates the time-efficient computation of the LBTLS estimator (Section 4).

3.2. Parameter estimation

At its core, the LBTLS estimator uses the Weighted Generalised Total Least Squares (WGTLS) method to obtain an estimate for the local rational model. This method is a natural extension of the least squares estimation procedure and yields a consistent estimate provided that the input-output noise covariances are known [30]. One of the advantages of the WGTLS method is that it works with the same linearised model equations as in (9) and thus the same Jacobian matrix (12) can be used to describe the linearised equations. Starting from $J$ in (12), the WGTLS method obtains an estimate for the local rational model by solving the following minimisation problem

$$\arg\min_J \|W (J - \hat{J}) C^{-1}\|_F^2$$

subject to $\hat{J} \theta = 0$ and $\theta^H \theta = 1 \quad (13)$

where $\| \bullet \|_F^2$ is the Frobenius norm, $W$ is a left weighting matrix, and $C$ is a square root of the column covariance matrix $C_{WJ}$ of the matrix pair $WJ$. It is crucial that $C$ is proportional to the square root of the true column covariance matrix

$$C = TC_{WJ}^{1/2} \quad \text{such that} \quad C^H C = C_{WJ} \quad (14)$$

where $T$ is an arbitrary orthogonal matrix. By doing so, the noise shaping present in $WJ$ can be exactly compensated for and, consequently, a consistent estimate can be obtained [18].

One of the main differences between the WGTLS method and the linear least squares estimation is the manner in which the parameter ambiguity is removed. Typically, this parameter ambiguity is solved by fixing one of the coefficients, mostly $d_{k0}$, to the identity matrix. Unfortunately, this constraint can lead to an ill-conditioned set of equations, especially if the true value of the fixed parameter is close to zero. The WGTLS method bypasses this issue by constraining the Euclidian norm of the parameters to one ($\theta^H \theta = 1$) such that every parameter can be taken into account for the estimation procedure.

In general, the choice of the weighting matrix $W$ does not affect the consistency of the WGTLS method, but mainly modifies the stochastic efficiency of the estimator. Similar to the Sanathanan-Koerner’s method, the LBTLS estimator improves the efficiency by iteratively updating $W$ with an approximation of the ML weighting [6] (Section 3.3). In each iteration, an improved estimate of the weighting is obtained resulting in a better estimate of the model parameters and so on. By implementing this iterative scheme, the LBTLS estimator is characterised by an efficiency which is close to the ML efficiency while maintaining
the favorable global minimisation properties of the WGTLS method [6]. The main difference with the Sanathanan-Koerner’s method is that in each iteration consistent estimates are obtained. This consistency property does not only ensure that the iteration can be stopped at any time, the number of iterations can also be kept limited: all simulation cases and practical applications suggest that 5 iterations are mostly sufficient to reach convergence.

3.3. Derivation of the left weighting matrix \( W \) and column covariance matrix \( C_{WJ} \)

Originally, the LBTLS estimator was developed in a global identification context where noise was imposed on both \( U \) and \( Y \). Although the local modelling context mostly considers the output-error framework, this subsection considers an errors-in-variables framework by taking into account both input noise \( V_U(k) \) and output noise \( V_Y(k) \) such that more general expressions for both \( W \) and \( C^H C \) can be obtained. During each iteration of the LBTLS estimator, the weighting \( W \) is adapted to approximate the ML weighting [22]. In the multi-dimensional case, it is impossible to apply the full ML weighting without violating the linearity condition of the parameters [18]. Adopting the common denominator structure, a frequency-dependent scalar weighting function \( w_j(\delta_r, \hat{\Theta}) \) can be constructed for each output \( Y_j \) using the covariance matrix of the input \( C_U(k) \in \mathbb{C}^{n_u \times n_u} \), the covariance matrix of the output \( C_Y(k) \in \mathbb{C}^{n_y \times n_y} \), and the cross-covariance between the input and output \( C_{YU} \in \mathbb{C}^{n_y \times n_u} \).

\[
w_j(\delta_r, \hat{\Theta}) = \sum_{i=1}^{n_u} \left( B^H_U(j, \hat{\Theta}) C_{u}^{(j)}(k + r) B(j, \hat{\Theta}) + \sum_{i=1}^{n_y} 2 \text{Real}\left[B^H_U(j, \hat{\Theta}) C_{u}^{(j)}(k + r) B(j, \hat{\Theta})\right] \right) \]

\[+ A^H(j, \hat{\Theta}) C_{Y}^{(j)}(k + r) A(j, \hat{\Theta}) - 2 \text{Real}\left(A^H(j, \hat{\Theta}) C_{YU}^{(j)}(k + r) B(j, \hat{\Theta})\right) \]

where \( \hat{\Theta} \) is the parameter vector obtained in a previous iteration, and \( \bullet^{[\alpha, \beta]} \) is the operator which selects the element belonging to row \( \alpha \) and column \( \beta \) of the corresponding matrix. As starting values for the iterative procedure, the parameter solution with \( w_j(\delta_r, \hat{\Theta}) = 1 \) is used. Observe that \( w_j(\delta_r, \hat{\Theta}) \) does not include the covariance contributions between the different outputs. This is inherent to the proposed common denominator structure in (9) and results in a loss of statistical efficiency compared with the ML efficiency.

The weighting matrix \( W(\delta_r, \hat{\Theta}) \) is constructed in such a way that \( WJ \) accommodates the proposed scalar weighting function \( w_j(\delta_r, \hat{\Theta}) \) in the following manner:

\[
WJ = \begin{bmatrix}
\sqrt{w_1^{-1}} U^T \otimes P_{1}^{nb} & \sqrt{w_1^{-1}} P_{1}^{nc} & \cdots & -\sqrt{w_1^{-1}} Y_1 P_{1}^{na} \\
\cdots & \cdots & \cdots & \cdots \\
\sqrt{w_n^{-1}} U^T \otimes P_{n}^{nb} & \sqrt{w_n^{-1}} P_{n}^{nc} & \cdots & -\sqrt{w_n^{-1}} Y_n P_{n}^{na}
\end{bmatrix}
\]  

(16)

Starting from the constructed \( WJ \) in (16), it is possible to derive an explicit expression for the column covariance matrix \( C_{WJ} \). The column covariance matrix evaluated at \( \Omega_{k+r} \) is defined as

\[
C_{WJ}(\Omega_{k+r}, \hat{\Theta}) = C^H C = \mathbb{E}(WJ^H(WJ))
\]

where \( \mathbb{E}(\bullet) \) is the expected value operator with respect to the noise realisation, and \( WJ \) is the part of \( WJ \) which exclusively depends on the additive noise present at the input and output

\[
WJ = \begin{bmatrix}
\sqrt{w_1^{-1}} V_1^T \otimes P_{1}^{nb} & 0 & \cdots & \cdots & -\sqrt{w_1^{-1}} V_1 P_{1}^{na} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\sqrt{w_n^{-1}} V_n^T \otimes P_{n}^{nb} & 0 & \cdots & \cdots & -\sqrt{w_n^{-1}} V_n P_{n}^{na}
\end{bmatrix}
\]

(18)
Combining (16) and (18) results in the following expression for $C^H C$

$$C_{WJ}(\Omega_{k+r}, \hat{\theta}) =
\begin{bmatrix}
|w_1|^{-1}\Phi_1 & -|w_1|^{-1}\Phi_2 \\
|w_2|^{-1}\Phi_1 & -|w_2|^{-1}\Phi_2 \\
\vdots & \\
|w_n|^{-1}\Phi_1 & -|w_n|^{-1}\Phi_n
\end{bmatrix}
$$

(19)

with

$$\Lambda = \mathbb{E}\left\{(V^T_U(k+r))^H \otimes (P^{\sigma_a})^H \cdot \left(V^T_U(k+r) \otimes P^{\sigma_a}\right)\right\}
= \mathbb{E}\left\{(V^T_U(k+r))^H V^T_U(k+r)\right\} \otimes (P^{\sigma_a})^H P^{\sigma_a}
= C^H_U(k+r) \otimes (P^{\sigma_a})^H P^{\sigma_a}
$$

(20)

$$\Phi_j = (C^{[j,j]}_{YU}(k+r))^T \otimes ((P^{\sigma_a})^H P^{\sigma_a})
$$

(21)

$$\Sigma_j = C^{[j,j]}_{Y}(k+r) ((P^{\sigma_a})^H P^{\sigma_a})
$$

(22)

To retrieve an expression as a function of the noise covariances, we applied in (20)-(22) the mixed-product property of the kronecker product $(\rho \otimes \sigma) \cdot (\tau \otimes \nu) = (\rho \tau) \otimes (\sigma \nu)$ [29]. $\cdot^{[j,j]}$ is the operator which selects the $j^{th}$ row from a matrix.

To ensure the consistency and stochastic efficiency, the construction of $w_j(\delta)$ and $C_{WJ}$ requires the exact knowledge of the true $C_U$, $C_Y$ and $C_{YU}$. Unfortunately, information about these quantities is not available a priori which can undermine the consistency of the LBTLS estimator. This lack of information can be mitigated by considering the following two aspects of the original problem formulation:

1. In the output-error framework, $C_U$ and $C_{YU}$ are equal to zero.
2. Similar to LPM and LRM, it makes sense to assume that, in every small local frequency window, the disturbing output noise is white for each output seperately: $C^{[j,j]}_{Y} = \sigma^2_{Yj}$, $C^{[j,j]}_{YU} = 0$ for ($i \neq j$).

Since $\sigma^2_{Yj}$ is unknown beforehand, an additional simplification is made which rewrites the output noise covariance matrix as $C_Y(k+r) = \sigma^2 I_{n_r}$ where $I_{n_r} \in \mathbb{C}^{n_r \times n_r}$ is the identity matrix, and $\sigma^2$ is the unknown power of the white noise in the local window. On the condition that the noise in the local window can be well approximated by this simplification, the LBTLS estimator remains consistent (see Appendix A) but suffers a loss in stochastic efficiency. Additionally, the exact knowledge of $\sigma^2$ is not necessary since it acts as a scaling factor which can be put in front of $WJ$ and $C_{WJ}$ and, therefore, does not influence the retrieved solution.

Transformation from the errors-in-variables framework to the output-error framework ($C_U = C_{YU} = 0$) greatly shortens the expressions which are eventually used by the LBTLS estimator to derive consistent local model parameters. First of all, the weighting matrix $w_j$ in (15) just has to compensate for the wrong noise shaping induced by the common-denominator part. Secondly, $\Lambda$ and $\Phi_j$ become zero and the column covariance matrix in (19) is simplified since it only considers noise on the common-denominator coefficients.
3.4. Calculation of uncertainty bounds

Aside from providing a consistent estimate of the FRF and the transient term, the LBTLS estimator allows the calculation of uncertainty bounds on the estimated properties without imposing heavy requirements on the Signal-to-Noise Ratio (SNR). The classical LRM, which uses the least squares estimation procedure, recommends a minimal SNR of 20 dB such that the bias error becomes sufficiently small [21] and a linearisation can be applied to approximate the covariance matrix of the output noise \( V_Y \) [11]. Earlier conducted research revealed that this requirement can be softened for the ML estimator in the errors-in-variables framework [31]. Only in frequency regions where both the input and output SNR are both very low (smaller than 3 dB), a deviation between the sample ML estimator and the Cramér-Rao lower bound (smallest possible stochastic uncertainty) can be observed. Luckily this situation hardly occurs in most measurement applications. Since the LBTL estimator belongs to the class of approximate ML estimators, a SNR of at least 3 dB for both the input and output in the same frequency region is a strict minimum to guarantee a correct estimation of the uncertainty bounds.

The uncertainty bounds on the estimated parameters can be calculated starting from the equivalent cost function \( V_{LBTLs} \) of the parameter estimation problem in (13)

\[
V_{LBTLs}(\theta) = \sum_{r=-\Omega}^{\Omega} e(\Omega_{k+r}, \theta) W(\delta_r, \hat{\theta}) [W^H(\delta_r, \hat{\theta}) W(\delta_r, \hat{\theta})]^{-1} \sum_{r=-\Omega}^{\Omega} e(\Omega_{k+r}, \theta) W(\delta_r, \hat{\theta}) e(\Omega_{k+r}, \theta)
\]

subject to \( \theta^H \theta = 1 \)

with

\[
e(\Omega_{k+r}, \theta) = W(\delta_r, \hat{\theta}) e(\Omega_{k+r}, \theta)
\]

\[
C_\varepsilon(\theta) = \sum_{r=-\Omega}^{\Omega} \theta^H C_{WJ}(\Omega_{k+r}, \hat{\theta}) \theta
\]

\[
r(\Omega_{k+r}, \theta) = C_\varepsilon^{-1/2}(\theta) e(\Omega_{k+r}, \theta)
\]

and \( \hat{\theta} \) is the parameter vector obtained in the last iteration of the LBTL estimator. The covariance matrix of the estimated parameters can be approximated as (\( \text{Re}[\bullet] \) takes the real part of \( \bullet \))

\[
\text{Cov}(\hat{\theta}) \approx (2 \text{Re}[\mathcal{J}^H \mathcal{J}])^{-1}
\]

provided that sufficient SNR (> 6 dB) is available and that the model orders \( n_A, n_B \) and \( n_C \) are chosen large enough such that only small model errors are present [6, Section 12.3.4]. \( \mathcal{J} \) is the Jacobian of (23) which requires the derivation of \( r(\Omega_{k+r}, \theta) \) as a function of \( \theta \). Analytical calculation of this derivative is impossible and can be well approximated by the construction of a so-called pseudo-Jacobian matrix \( \mathcal{J}_+ \) [32]

\[
\mathcal{J}_+(\Omega_{k+r}, \hat{\theta}) = C_\varepsilon(\hat{\theta})^{-1/2} \left( \frac{\partial e(\Omega_{k+r}, \hat{\theta})}{\partial \hat{\theta}^{[i]}} - \frac{1}{2} \frac{\partial C_\varepsilon(\hat{\theta})}{\partial \hat{\theta}^{[i]}} C_\varepsilon(\hat{\theta})^{-1} e(\Omega_{k+r}, \hat{\theta}) \right)
\]
Substituting $\mathcal{F}$ with
\[
\begin{pmatrix}
\mathcal{F}^+(\Omega_{k-\tau/2}, \hat{\theta}) \\
\vdots \\
\mathcal{F}^+(\Omega_{k+\tau/2}, \hat{\theta})
\end{pmatrix}
\]
greatly facilitates the analytical calculation of the covariance matrix of the estimated parameters.

Starting from the calculated Cov($\hat{\theta}$), it is feasible to deduce the corresponding uncertainty on the FRF estimate. This derivation starts from a linearisation of the parametric FRF $G(\Omega_{k+r}, \hat{\theta})$ around the expected value $E(\hat{\theta})$
\[
G(\Omega_{k+r}, \hat{\theta}) \approx G(\Omega_{k+r}, E(\hat{\theta})) + \frac{\partial G(\Omega_{k+r}, \theta)}{\partial \theta} (\hat{\theta} - E(\hat{\theta}))
\]
By applying the vec($\bullet$) operator on $G$, and by combining it with the following property of the Kronecker product $\text{vec}(\rho \sigma \tau) = (\tau^T \otimes \rho) \text{vec}(\sigma)$, an expression for the covariance matrix of the vectorised FRF matrix is obtained
\[
\text{Cov}(\text{vec}(G(\Omega_{k+r}, \hat{\theta}))) \approx \left( \mathbf{I}_{n_\theta} \otimes \frac{\partial G(\Omega_{k+r}, \theta)}{\partial \theta} \right) \text{Cov}(\hat{\theta}) \left( \mathbf{I}_{n_\theta} \otimes \frac{\partial G(\Omega_{k+r}, \theta)}{\partial \theta} \right)^T
\]
where we additionally approximated the unknown expected value $E(\hat{\theta})$ with the estimated parameters $\hat{\theta}$. The variances of $G^{\mu,\ell}$ can subsequently be derived from (31) by merely considering the diagonal values of Cov(vec($G(\Omega_{k+r}, \hat{\theta}$)).

3.5. Noisy input, noisy output observations of systems operating in closed-loop

Until now, the assumption has been often made that the output-error framework was valid and therefore the input was not corrupted with noise. However, feedback is present in any experimental setup where a dynamic interaction exists between the non-ideal actuator (e.g. mechanical shaker) and the mechanical structure, resulting in a dependency of the actual input of the system on both the actuator and system dynamics. For this reason, the input of the system should be measured and consequently the output-error framework is no longer valid.

Whenever the input data is corrupted with noise and/or if feedback is present, ensuring consistency with the earlier proposed white noise approximation (Section 3.3) becomes difficult since all the covariance matrices $C_U, C_Y$ and $C_{UY}$ should be known accurately. To circumvent this difficulty, a reference signal $R(k) \in \mathbb{C}^{n_u \times 1}$, usually the signal stored into the arbitrary waveform generator, should be available to model the system indirectly [20]. Instead of modelling the system directly from the noisy input $U$ to the noisy output $Y$, both $U$ and $Y$ are modelled simultaneously from the known reference $R$ by replacing (1) with
\[
Z(k) = G_r(\Omega_k) R(k) + T_r(\Omega_k) + V_Z(k)
\]
where
\[
Z(k) = \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix}, \quad G_r(\Omega_k) = \begin{bmatrix} G_{ry}(\Omega_k) \\ G_{ru}(\Omega_k) \end{bmatrix}, \quad T_r(\Omega_k) = \begin{bmatrix} T_{ry}(\Omega_k) \\ T_{ru}(\Omega_k) \end{bmatrix}, \quad V_Z(k) = \begin{bmatrix} V_Y(k) \\ V_U(k) \end{bmatrix}
\]
After identifying $G_r$ and $T_r$, with the earlier described local modelling techniques, the FRF $G$ of the system can be derived from $G_r$ as follows
\[
G(\Omega_k) = G_{ry} G_r^{-1}
\]
while the covariance matrix of $G$ can be deduced from the covariance matrix of $G_{rz}$ in the following manner [6, Section 7.2.7.1]

$$\text{Cov}(\text{vec}(G)) \approx (G_{ru}^T \otimes [I_{n_y}, -G]) \text{ Cov}(\text{vec}(G_{rz})) (G_{ru}^T \otimes [I_{n_y}, -G])^H$$ (35)

Introducing the known reference signal in the local modelling techniques allows us to efficiently convert the closed-loop identification problem to the output-error framework. As shown in (34) and (35), only a marginal post-processing step is required to deduce from this augmented modelling in (32) a consistent estimate of the FRF matrix and uncertainty bounds.

4. Numerically efficient computation

The global minimiser of (13) is obtained via the Generalised Singular Value Decomposition (GSVD) of the matrix pair $(WJ, C)$ [18, 33]. However, this GSVD algorithm starts with the computation of the upper triangular matrix $R_{WJ}$ of the QR factorisation of $WJ$ which is computationally expensive whenever a large amount of inputs and outputs is considered. We propose to explicitly take advantage of the structure of $WJ$ to counteract the dimensional explosion (Figure 3). Instead of computing one QR decomposition of the whole matrix $WJ$, as done in a generic GSVD implementation, $n_y$ different QR decompositions are performed separately for each output $Y_j$. Each upper triangular matrices $R_j$ belonging to output $Y_j$ can be divided into three parts which describe the projection of the numerator coefficients (top triangle ▲), the denominator coefficients (bottom triangle ▼) and the mapping between these numerator and denominator coefficients (square ■). Since the numerator coefficients of the different outputs are modelled independently, ▲ and ■ can be used without further modification for the construction of $R_{WJ}$. Only for the denominator
part, one additional QR decomposition needs to be performed on the stacked lower parts to compute the final projection $R_{den}$ belonging to the denominator coefficients.

To analyse the computational complexity of the proposed method, the number of floating point operations (flops) needed to compute $R_{WJ}$ is compared. The computational complexity of the QR decomposition is characterised by $2mn^2$ flops for a $m \times n$-matrix [34]. Straightforward application of the QR decomposition on $WJ$ would then result in

$$2 (2n_\Delta + 1) n_y \left[ (n_A + 1) + [(n_B + 1) n_u + (n_C + 1)] n_y \right]^2 = O(n_y^3) \text{ flops}$$  \hspace{1cm} (36)$$

while our proposed method results in

$$2 n_y (2n_\Delta + 1) \left[ (n_A + 1) + (n_B + 1) n_u + (n_C + 1) \right] n_y^2 = O(n_y) \text{ flops}$$  \hspace{1cm} (37)$$

Analysing (36) and (37) shows that the proposed method effectively transforms the computational complexity of the QR decomposition from a cubic dependence on $n_y$ to a linear one.

If there is no noise present at the input (output-error framework) then a second measure can be applied to improve the numerical efficiency. In that case, the matrices $\Lambda$ and $\Phi$ of the column covariance matrix $C_{WJ}$ in (19) are zero. Consequently only a GSVD of the common denominator part is necessary such that the dimensional explosion on the number of inputs and outputs can be further reduced. Computing the QR decomposition of $WJ$ makes that the projection onto the null-space of the noiseless part of the matrix is already performed and consequently the block structure of the resulting $R_{WJ}$ matrix can be used to increase the efficiency of the computation with respect to the number of outputs [30]. The original model equations (12) can be rewritten using the QR decomposition of $WJ$ as follows

$$WJ \theta = Q \begin{pmatrix} R_{num} & R_{map} \\ 0 & R_{den} + \Delta R_{den} \end{pmatrix} \begin{pmatrix} \theta_{num} \\ \theta_{den} \end{pmatrix} = 0$$  \hspace{1cm} (38)$$

where $R_{num}$, $R_{map}$ are, respectively, the numerator and mapping part of $R_{WJ}$ as shown in Fig. 3 and $\theta_{num}$, $\theta_{den}$ are, respectively, the numerator and denominator coefficients. Since the input is not perturbed with noise, only the part of the denominator coefficients $R_{den}$ is affected by the output noise via $\Delta R_{den}$. Therefore, the GSVD should only be applied on the matrix pair ($R_{den} + \Delta R_{den}$, $\sum_{j=1}^{n_y} [w_j]^{-1} \Sigma_j$) to retrieve an estimate for the denominator coefficients $\theta_{den}$. Afterwards, the numerator coefficients $\theta_{den}$ can be obtained by manipulating the first row of (38) in the following manner

$$\theta_{num} = -(R_{num})^{-1} R_{map} \theta_{den}$$  \hspace{1cm} (39)$$

By combining the above mentioned measures, a computationally efficient scheme is obtained which allows us to obtain estimates for systems with a large number of input and outputs, and this within a reasonable time frame.

5. Simulation example

This section evaluates the performance of the proposed LBTLS estimator on a simulation example with respect to currently existing local modelling techniques [8, 11]. Three different estimators are compared with each other: the LPM, the iterative LRM employing the Sanathanan-Koerner method, and the iterative
LBTLS estimator. For the simulation set-up, the configuration displayed in Fig. 1 is used in which $G$ possesses the dynamic behaviour of a lightly damped mechanical system with the following model structure

$$G^{[j, i]}(j\omega_k) = \sum_{n=1}^{N_p} p_n^2 \frac{r_{[j, i]}^n}{j\omega_k - p_n} + \frac{r_{[j, i]}^n}{j\omega_k - \overline{p}_n}$$

where $N_p$ is the number of natural modes in the system, $p_n$ are the pole locations and $r_n \in \mathbb{C}^{n_y \times n_u}$ are the residue matrices. All the poles have the same relative damping equal to $1.25 \times 10^{-3}$ and are equally spaced in the simulated frequency window. Observe that the residue matrix of each mode is multiplied with $p_n^2$ or $\overline{p}_n^2$ such that an FRF with a high dynamic range is obtained. Furthermore, the residue matrices are chosen random to ensure that each natural mode is expressed differently for all input-output combinations. To not overcomplicate the simulation example, the system transient contribution $T$ is not included in the model (the measurement example in Section 6 does model $G$ and $T$ simultaneously). The noise perturbation $V_f(k)$ at the output has been added in such a way that a SNR of 40 dB is obtained at each frequency. In this section, three different aspects are evaluated: (i) the influence of the number of iterations on the estimated FRF (Section 5.1), (ii) the comparison of the LBTLS estimator with the traditional LPM and LRM (Section 5.2), and (iii) application of the LBTLS estimator to a large-scale MIMO system of order 100 with 100 inputs and 100 outputs (Section 5.3).
Figure 5: Applying the proposed LBTLS estimator on a system with 2 inputs and 3 outputs for varying iteration counts reveals that the largest gain in accuracy is obtained in the boundary regions. The RMSE of the estimated FRF $\hat{G}^{[1,2]}$ without iterating (blue •) and 5 iterations (green •) is compared with the reference standard deviation of the FRF (grey −).

5.1. Influence of iterating within the LBTLS estimator

To properly evaluate the impact of the number of iterations, we compare the estimate without iterations with the estimate after 5 iterations. For this purpose, the LBTLS estimator is applied to the MIMO system in (40) with 2 inputs, 3 outputs and $N_p = 9$ (Fig. 4). From the input-output data, the FRF is estimated in the frequency band [5 Hz, 1000 Hz] via the LBTLS estimator with $n_A = n_B = 1$. $n_A$ and $n_B$ are chosen such that at least the contribution of one pole (or zero) could be modelled. Visual inspection of the results with $n_A = n_B = 2$, showed that unwanted pole-zero cancellations occurred in the regions with small dynamic variations which in turn increased the modelling error. Furthermore, $n_{\Delta \Omega}$ is chosen such that the degrees of freedom of the local model, defined as $2n_{\Delta \Omega} + 1 - n_\theta$, are equal to 8 [6]. Every method has been applied to $n_{\text{exp}} = 100$ different Monte-Carlo realisations of the noise perturbation.

To assess the goodness of fit of the estimated model, we compare the reference standard deviation derived from the exact $C_Y(k)$ with the Root Mean Square Error (RMSE) of the estimators, defined as

$$\text{RMSE}(\hat{G}^{[j,i]}(j\omega_k)) = \sqrt{\frac{1}{n_{\text{exp}}} \sum_{l=1}^{n_{\text{exp}}} \left| \frac{\hat{G}^{[j,i]}_{l}(j\omega_k) - G^{[j,i]}_{l}(j\omega_k)}{G^{[j,i]}_{l}(j\omega_k)} \right|^2}$$

where $\hat{G}_{l}(j\omega_k)$ is the acquired transfer function estimate for the $l^{th}$ Monte-Carlo realisation. The reference standard deviation (−) can be computed using the noiseless regressor matrix of the LPM and the known output noise covariance matrix $C_Y$ [6, Section 7.2.2.4]. For clarity, only one input-output combination is shown which was selected randomly from the FRF matrix (Fig. 5). A good fit is characterised by a RMSE which approximates well the reference standard deviation. Remark that this standard deviation of $G^{[1,2]}$ contains resonances at 230 Hz, 510 Hz, 633 Hz and 749 Hz which corresponds exactly with some of the anti-resonances of $G^{[1,2]}$ (Fig. 4). These resonances are an unwanted artefact of the division of the standard deviation with $G^{[1,2]}$ which, due to the lower values of $G^{[1,2]}$ in the anti-resonance regions, can potentially create misleading resonances. However, these artefacts do not affect the interpretation of the results. Comparing the RMSE of the estimated FRF over the different Monte-Carlo realisations reveals that
Figure 6: The RMSE of the estimated FRF $\hat{G}_{1,2}$ using LPM (black •) and LRM with 5 iterations (red •) is compared with the FRF obtained using the LBTLS estimator with 5 iterations (green •). Additionally, the reference standard deviation of the FRF is shown (grey –).

the LBTLS estimator without iteration already provides an accurate estimate of the FRF. Iterating results in a gain of accuracy which is especially visible at the boundaries of the considered frequency spectrum.

5.2. Comparison with the traditional LPM and LRM

By adopting the same simulation conditions as in Section 5.1, the performance of both the LPM [8] and LRM [11] can be analysed and compared with the earlier obtained LBTLS estimate (Fig. 6). In the case of the LPM a 4th order polynomial is used with 8 degrees of freedom, while the same properties are used for the LRM and LBTLS estimator (see Section 5.1). Furthermore, we applied the Sanathanan-Koerner’s method (5 iterations) to the LRM such that a fair comparison can be made with the iterative LBTLS estimator.

As expected, the LPM provides an accurate estimation of the FRF in the anti-resonances and in the transition regions from these anti-resonances to the resonances (Fig. 6a). However, a poor quality estimate is obtained in the resonance regions due to the limited modelling capacity of the polynomial model. Inspection of the FRF reveals that the number of frequency points present in the 3dB bandwidth of the resonances ranges from 1 to 11. However, the LPM requires at least 7 frequency points to ensure that the approximation error remains sufficiently low [13]. The LRM on the other hand (Fig. 6b), greatly improves the modelling of the
Table 1: Cost function for the different methods applied to the simulation example. The number next to the methods designates the number of iterations used during the estimation.

<table>
<thead>
<tr>
<th>Method</th>
<th>$V_{cost}$</th>
<th>$n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LPM</td>
<td>2.71e3</td>
<td>5</td>
</tr>
<tr>
<td>LRM-1</td>
<td>0.71</td>
<td>4</td>
</tr>
<tr>
<td>LRM-5</td>
<td>1.12</td>
<td>4</td>
</tr>
<tr>
<td>LBTL-1</td>
<td>1.06</td>
<td>4</td>
</tr>
<tr>
<td>LBTL-5</td>
<td>0.64</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 7: The STandard Deviation (STD) of the estimated FRF $\hat{G}[1, 2]$ using LPM (black •) is compared with the LBTLS estimator with 5 iterations (green •). The reference standard deviation of the FRF for one realisation is also depicted (grey −).

resonances but has more difficulties to correctly estimate the FRF in the flat regions (0 - 0.25 kHz) and at the boundaries. This behaviour is presumably caused by the removal of the parameter ambiguity which fixes one of the denominator coefficients to the identity matrix (see Section 3.2). A consequence of this choice is the disability to completely remove the contribution of the denominator polynomial in the anti-resonances and flat regions, which adversely influences the accuracy of the estimate. This disability is removed by the LBTLS estimator, resulting in a local modelling technique which performs better over the whole frequency range compared with LPM and LRM.

To further evaluate the performance of the different methods over the whole frequency range, the calculated values of the following cost function are compared (Table 1)

$$V_{cost} = \frac{1}{F} \sum_{k=1}^{F} \text{vec}(\hat{G}(j\omega_k) - G(j\omega_k))^H C_{\text{vec}(G)}^{-1}(j\omega_k) \text{vec}(\hat{G}(j\omega_k) - G(j\omega_k))$$

(42)

where $F$ is the total number of frequencies, $\hat{G}$ is the mean of $\hat{G}_l$ over the different Monte-Carlo realisations and $C_{\text{vec}(G)}$ is the reference covariance matrix of the stacked FRF matrix. From Table 1, it can be concluded that the LPM is unfavorable to model resonant systems due to the high cost function value. The local rational modelling techniques have a comparable performance to model these resonant systems. Application of the LBTLS estimator with 5 iterations results in the lowest value of $V_{cost}$. Remark that iterating with the LRM increases the cost function value as has been earlier reported in [21].
Another important feature of the LBTLS estimator is the capability to accurately estimate uncertainty bounds on the acquired FRF (Section 3.4). To verify the proposed derivation of uncertainty bounds, we compared the Standard Deviation (STD) obtained with the LPM and the LBTLS estimator for one single Monte-Carlo realisation (Fig. 7). Again, the LBTLS estimator improves the overall estimation of the uncertainty bounds compared to those obtained with the LPM.

5.3. Application to a large-scale MIMO system

In this section, the effectiveness of the improved numerical scheme (Section 4) is illustrated on a large-scale MIMO system of order 100 with 100 inputs and 100 outputs. The same simulation configuration as in (40) is used where we set \( N_p = 100 \) such that the LBTLS estimator could be challenged to describe a complex lightly-damped system. The LPM (2\textsuperscript{nd} order polynomial), the LRM \((n_A = n_B = 1)\) and LBTLS estimator \((n_A = n_B = 1)\) are applied to the noisy input-output data with 8 degrees of freedom in the local model (Fig. 8).

To verify the performance of these methods in estimating the non-parametric FRF matrix, we visualise the results obtained with the LPM and the LBTLS estimator for one arbitrary transfer function \( \hat{G}_{[42,17]} \) (the LRM had a similar performance as the LBTLS estimator). Unlike the LBTLS estimator, Fig. 8 shows that the LPM cannot adequately model the resonant behaviour of the system. Due to the high number of inputs and outputs, the local window size increases significantly to identify all the coefficients. As a result, the FRF matrix needs to be modelled over a broader dynamic range which is challenging for polynomials. One solution would be to increase the order of the polynomials because this generally improves the modelling capabilities of the LPM. Unfortunately, increasing the order negatively impacted the estimate since a larger order resulted in a wider local window in which the dynamic behaviour varied even more.

Comparing the computational cost of the methods on an Intel Xeon E5-2630 v3 @2.4 GHz (Table 2), reveals that the LBTLS estimator increases the computation time compared with the LPM and the LRM. Luckily, this increase can be confined by implementing the numerically efficient scheme as outlined in Section 4. Putting these computation times in perspective indicates that the LBTLS estimation of a consistent local rational model comes with a cost: real-time FRF identification will not be possible, even for small-scale systems where the computation time for one iteration and one frequency is in the order of 10 milliseconds. When choosing an appropriate method for an application, the user needs to carefully balance his/her needs concerning efficiency and accuracy:

- Spectral analysis methods, such as the \( H1 \) estimator [2], are very computationally efficient but lack the superior transient suppression of the local modelling techniques [6].

- Local modelling techniques improve the transient suppression but become less efficient as the accuracy increases.

6. Measurements on the tailplane of a glider

Applying the proposed technique on a realistic measurement example is indispensable to illustrate the (dis)advantages in an experimental setting. Therefore, the resonant behaviour of the tailplane of a glider is characterised using the previously described LBTLS estimator. In this section, we describe the measurement setup and compare the results obtained with the proposed technique, the LPM and the LRM.

The system under test is the tailplane of a glider which is connected to a wall via its central mounting points (Fig. 9). The tailplane is excited by two mini-shakers (B&K 4810) and the responses are measured
Figure 8: The error of the estimated FRF $\hat{G}^{[42,17]}$ using LPM (black •) and the LBTLS estimator with 5 iterations (green ×) is compared with the true FRF (grey −). Additionally, a zoomed-in view between 800 Hz and 900 Hz shows the estimated FRF using LPM (black ×) and the LBTLS estimator (green ×).

at 5 different locations using impedance heads (B&K 8001). Both the generation (HPE 1445A) and recording (HPE 1430A) of the different signals is managed by the VXI measurement system which internally synchronises these processes at the same sampling frequency.

Unlike the simulation example, the output-error framework is invalid due to feedback which is caused by the dynamic interaction between the shakers and the glider’s tailplane. To reveal this unknown dynamic interaction, two impedance heads are directly connected to the mini-shakers to record the signals which are truly applied to the system. Applying the local modelling technique directly from the input force sensors (1 and 2) to the output acceleration sensors (3, 4 and 5) would result in a bias since both the inputs and outputs are distorted with noise. In contrast, performing the modelling from the known reference signals to both the noisy inputs and noisy outputs effectively transforms the closed-loop identification problem to the output-error framework and eliminates this bias altogether (Section 3.5).

Using the described measurement setup, filtered white noise with a cutoff frequency of 100 Hz was applied as an excitation to both the mini-shakers. Until now, only noise was considered as an external disturbance. However, no system is completely linear, meaning that the system under test will generate
Table 2: Average computation time needed to estimate a local model at one single frequency for a MIMO system with 100 inputs and 100 outputs. In case of the LBTLS estimator and the LRM, the time for performing one single iteration is displayed.

<table>
<thead>
<tr>
<th></th>
<th>LPM</th>
<th>LRM</th>
<th>Efficient LBTLS</th>
<th>Traditional LBTLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average computation time</td>
<td>0.041 s</td>
<td>1.84 s</td>
<td>4.29 s</td>
<td>1.5 hours</td>
</tr>
<tr>
<td>Size regression/Jacobian matrix</td>
<td>((2n_\Omega + 1) n_y \times (n_B + 1) n_u + (n_C + 1))</td>
<td>((2n_\Omega + 1) n_y \times [(n_B + 1) n_u + (n_C + 1)] n_y)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 9: The VXI measurement setup makes it possible to identify the resonant behaviour of the tailplane of a glider.

nonlinear distortions which, in turn, will further increase the output covariance \(C_Y\) [20]. We deliberately applied large forces onto the tailplane of the glider to make sure that a fair amount of nonlinear distortion was present in the measurement data. By doing so, we could verify how the proposed technique behaves in an experimental setting where the signal to distortion ratio is lower than 20 dB. A data set was obtained with a total measurement time of 108 s using a sampling frequency of 305 Hz. To generate a reference estimate, the LPM was applied to this complete dataset using a 4th order polynomial for both \(G\) and \(T\).

The main goal of local modelling techniques in general is to generate a reliable estimate when the amount of available data is limited. To this purpose, we restricted the length of the data set to one tenth...
of the total measurement time and subsequently applied the LPM (4th order polynomial), the LRM \(n_A = n_B = n_C = 1\) and the LBTLS estimator \(n_A = n_B = n_C = 1\) on this shortened data record (see Fig. 10 for a comparison between LPM and LBTLS). Analysing the obtained results with the cost function defined in (42) reveals that the LPM captures less well the dynamic behaviour than LBTLS. Again, the LRM and LBTLS have a comparable performance in modelling the resonant behaviour of the system. To further evaluate the performance of the different techniques (LPM, LRM and LBTLS), we zoomed in on a resonance and anti-resonance of \(G^{[4,1]}\) (Fig. 11). This zoomed-in view reveals that LPM behaves poorly in estimating the resonance, while LBTLS and LRM improve this estimation. In the other regions, there is no significant difference between the three methods.

The main differences between both the LPM and the LBTLS estimator can be observed in the estimation of the uncertainty bounds (Fig. 12):
The non-parametric local modelling technique presented in this paper allows one to accurately estimate the FRF and corresponding uncertainty bounds for large-scale multiple-input, multiple-output lightly-damped mechanical systems. The LBTLS estimator provides a consistent estimate of the non-parametric FRF which ensures the derivation of accurate uncertainty bounds.

7. Conclusions

The non-parametric local modelling technique presented in this paper allows one to accurately estimate the FRF and corresponding uncertainty bounds for large-scale multiple-input, multiple-output lightly-damped mechanical systems. The LBTLS estimator provides a consistent estimate of the non-parametric FRF which ensures the derivation of accurate uncertainty bounds.
damped systems using arbitrary excitations. This technique uses the bootstrapped total least squares estimator to obtain an unbiased estimate of the common-denominator rational FRF matrix which improves the consistency of currently existing rational local modelling techniques at the expense of increasing the computational complexity. Furthermore, the dimensional explosion for large-scale systems can be diminished by taking explicit advantage of the common-denominator structure. The proposed technique has been successfully applied to lightly-damped mechanical systems, both in simulation as in practice, and demonstrates that our technique is an effective alternative for currently existing local modelling techniques.

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Appendix A. Consistency LBTLS under white noise approximation

In this appendix, we investigate the influence of the white noise approximation on the consistency of the LBTLS estimator in the output-error framework. To prove consistency of an estimator, it suffices to prove that the expected value of the cost function is minimal in the true model parameters \( \theta_0 \). Since the cost function \( V_{LBTLS}(\theta) \) is a quadratic function of the measurements \( U \) and \( Y \), it can be split into two parts

\[
E[V_{LBTLS}(\theta)] = E[V_{LBTLS}(\theta, U_0, Y_0)] + E[V_{LBTLS}(\theta, V_Y)] \tag{A.1}
\]

where \( E[V_{LBTLS}(\theta, U_0, Y_0)] \) depends entirely on the noiseless input-output data \( U_0, Y_0 \) and \( E[V_{LBTLS}(\theta, V_Y)] \) only contains the contribution of the output noise \( V_Y \). From (A.1), it follows that the LBTLS estimator is consistent if \( E[V_{LBTLS}(\theta, V_Y)] \) is a \( \theta \)-independent constant. To verify this parameter dependency, we elaborated an explicit expression for \( E[V_{LBTLS}(\theta, V_Y)] \) as a function of \( \theta \). Considering the output-error framework, the original formulation of the cost function (23) can be simplified to the following expression for \( E[V_{LBTLS}(\theta, V_Y)] \)

\[
E_{\hat{\theta}_den}[V_{LBTLS}(\theta, V_Y)]=\frac{\sum_{r=-n_M}^{n_M} \sum_{l=-n_M}^{n_M} \sum_{j=1}^{n_y} |w_j(\hat{\delta}_r, \hat{\theta}_{den})|^{-1} A(\hat{\delta}_r, \hat{\theta}_{den}) C^T_{Y}[j](k+r) |\theta_{den}|^2}{\sum_{l=-n_M}^{n_M} \sum_{j=1}^{n_y} |w_j(\delta_r, \hat{\theta}_{den})|^{-1} \theta_{den}^H \Sigma_j \theta_{den}} \tag{A.2}
\]

where \( \theta_{den} \) is the subset of \( \theta \) which contains the denominator coefficients. Further pairwise multiplication of the elements in the numerator, simplifies (A.2) to

\[
E[V_{LBTLS}(\theta, V_Y)] = \frac{\sum_{r=-n_M}^{n_M} \sum_{l=-n_M}^{n_M} \sum_{j=1}^{n_y} |w_j(\delta_r, \hat{\theta}_{den})|^{-1} |A(\delta_r, \theta_{den})|^2 C^T_{Y}[j](k+r) |\theta_{den}|^2}{\sum_{l=-n_M}^{n_M} \sum_{j=1}^{n_y} |w_j(\delta, \hat{\theta}_{den})|^{-1} \theta_{den}^H \Sigma_j \theta_{den}} \tag{A.3}
\]

Applying the white noise approximation on \( \Sigma_j \) and \( w_j \)

\[
\theta_{den}^H \Sigma_j \theta_{den} = \sigma^2 \theta_{den}^H (P^{na})^H P^{na} \theta_{den} = \sigma^2 |A(\delta_r, \theta_{den})|^2 \tag{A.4}
\]

\[
w_j(\delta_r, \hat{\theta}_{den}) = \sigma^2 |A(\delta, \hat{\theta}_{den})|^2 \tag{A.5}
\]

and substituting (A.4),(A.5) in (A.3) results in

\[
E[V_{LBTLS}(\theta, V_Y)] = \frac{\sum_{r=-n_M}^{n_M} |A(\delta_r, \theta_{den})|^2 \sum_{j=1}^{n_y} \frac{C^T_{Y}[j](k+r)}{\sigma^2} |\theta_{den}|^2}{\sum_{l=-n_M}^{n_M} |A(\delta_r, \theta_{den})|^2 \sum_{j=1}^{n_y} |\theta_{den}|^2 n_u} \tag{A.6}
\]

Analysing the expression in (A.6), shows that \( E[V_{LBTLS}(\theta, V_Y)] \) only becomes \( \theta \)-independent if we assume that the noise in the local window can be well approximated by white noise.
References


