Consistent multi-input modal parameter estimators in the frequency domain

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Abstract

In this paper, consistent multiple-input frequency-domain estimators will be presented based on a right matrix-fraction description of the frequency response function. This matrix-fraction description leads to a fast algorithm in the same way as the least-squares complex frequency-domain estimator (LSCF). The use of multiple inputs simultaneously in the estimation (so-called polyreference estimation) has the advantage that it allows to separate closely-spaced modes.

The main drawback of the LSCF estimator is that it is theoretically inconsistent, i.e. the estimates do not converge to the true values when the number of measurements increases to infinity. However, the noise information available from most measurements can be used to construct a maximum likelihood-like weighting for the LSCF estimators, giving consistent estimates. The results are fast, polyreference, and consistent weighted generalised total-least-squares (WGTLS) estimators. The iterative quadratic maximum likelihood (IQML) estimator is practically consistent for high signal-to-noise ratios, with the additional advantage that it yields clear stabilisation charts.

The performance of the presented WGTLS and IQML estimators is evaluated by means of ground vibration test data and demonstrated on flight flutter test data.

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1. Introduction

The modal parameters (resonant frequencies $f_r$, damping ratios $\zeta_r$, mode shapes $\psi_r$, and modal participation factors $\mathbf{L}_r = Q_r \psi_r$ with $Q_r$ the modal scaling) are global properties of a structure [1, 2]. This implies that there is only one answer for every modal parameter. Ideally a modal parameter estimator should thus yield only one estimate for every parameter. If multiple frequency response functions (FRF) are measured (e.g. at different locations of the structure) and processed independently then multiple answers are obtained for every parameter. All estimates will differ by a small amount. The estimates are said to be calculated in a local way, i.e. they depend on the measurement location. Global modal parameter estimators combine all FRF measurements in a simultaneous estimation algorithm, enforcing a global estimate. A drawback of this global approach is that data inconsistencies are not immediately obvious to the user. If the system is altered between measurements (e.g. by the changing mass-loading effect due to roving accelerometers), then an independent estimate of every FRF (local estimate) would reveal any discrepancies between the estimated modal parameters. The inconsistencies between the different measurements would be immediately clear. A global estimator would combine all (incompatible) measurements to yield only one but unreliable estimate, without it being obvious that it is unreliable. Still, any serious model validation should be able to reveal the problems with such a data set.

The earliest advances in the global estimation of modal parameters were in the late nineteen seventies and eighties, with the least-squares complex exponential (LSCE) estimator [3], the Ibrahim time domain (ITD) method [4], the Eigensystem realisation algorithm (ERA) [5], and the direct parameter estimation [6], all time-domain estimators. These estimators curve-fit the impulse response function which is often obtained by inverse Fourier transforming the measured FRF. Time-domain identification is very popular and its development is still being pursued (see e.g. [7]).

Vold et al. [8] introduced a polyreference implementation of the LSCE estimator. They defined polyreference as the use of FRFs from multiple
exciter locations (multiple references) simultaneously in the modal parameter estimation process (and we adhere to that definition). In doing so, they were able to obtain a more accurate modal model and to determine multiple roots or closely-spaced modes of a structure. Since then, different polyreference time-domain and frequency-domain estimators have been developed (see e.g. [1]).

Contrary to time-domain estimators, frequency-domain estimators directly curve-fit the measured FRF. Early advances were the frequency domain direct parameter estimation [9] and the rational-fraction polynomial (RFP) based methods [10, 11, 12]. In [1], a unified framework was formulated for these estimators. Recent advances based on the RFP method are the single reference total-least-squares based estimators [13], the polyreference least-squares complex frequency-domain (LSCF) estimator, commercially known as the PolyMAX estimator [14], and a Z-domain version of the total-least-squares estimator [15].

In frequency-domain identification, modal parameter estimators are often based on the minimisation of a quadratic cost function. The FRF model can be a single-reference or a polyreference rational polynomial model. In both approaches the denominator is forced to be common to all FRFs. This denominator is a scalar polynomial for the single-reference model and a matrix polynomial for the polyreference method. Therefore, both single-reference and polyreference estimators yield global estimates for the system poles (resonant frequencies and damping ratios).

For the single-reference estimator also the residues estimates corresponding to these poles are global. However, the decomposition of these residues into the mode shapes and participation factors is not unique. The mode shapes and participation factors are thus estimated in a local way. This estimation step is generally done in a least-squares sense by taking the left (mode shape) and right (participation factor) singular vector corresponding to the largest singular value. The mode shapes and participation factors obtained in this way are proportional up to a scaling constant, and thus they form a residue matrix of rank one, in agreement with modal analysis theory [2]. However, the quality of the FRF fit based on the modal parameters (after the SVD) is worse compared with the polynomial model synthesised fit. The freedom of the single-reference estimator (by not imposing a rank one constraint on the residues) gives more freedom to the estimator and allows for a better fit of the FRF [16]. An often used alternative is the Least-Squares Frequency-Domain (LSFD) estimator (see e.g. [17]). This LSFD approach
is equally useful for the single-reference estimator and yields often good results compared with the SVD mentioned before. The calculation of the mode shapes can then be limited to only those corresponding to the poles selected from the stabilisation diagram, and the FRF fit is in general better than with the SVD approach.

The polyreference model estimates the system poles, mode shapes, and modal participation factors in a global way. Both the poles and the participation factors are obtained from the denominator matrix polynomial. The mode shapes can be found from the numerator polynomial or using the LSFD estimator (as for the single-references model). The main advantage of the polyreference approach is that a rank-one constraint on the residue matrices is implicit in the model, thus no accuracy is lost when transforming the polynomial model to the modal model (since no SVD or other estimator is needed to calculate the mode shapes from the residues). Moreover, the polyreference estimator (on multi-input data) can separate closely-spaced modes [8], which is very interesting for the testing of (large) aircraft structures [18]. The main drawback of the right matrix-fraction description is that the number of estimated modes is always a multiple of the number of measured inputs. Too high a model order is necessary to eliminate bias in the estimates (due to modelling errors), and thus a stabilisation chart is needed to remove the spurious poles.

Besides the clear stabilisation diagrams, the speed of the LSCF and its ability to handle large datasets has contributed to the widespread use of the LSCF in the modal analysis community [14, 19]. An important drawback, however, is that the LSCF estimator is not consistent, because it is a linear least-squares estimator [20, p. 200]. It is only consistent in the hypothetical noiseless case, or when the denominator polynomial is equal to 1. This latter requirement is obviously never fulfilled for practical modal analysis applications. Although consistency is an asymptotic property and may seem of little practical interest at first sight, it does imply that the uncertainty on the estimates decreases when more data is gathered. For inconsistent estimators, always some error remains, even if they are (asymptotically) unbiased [20, p. 455].

One challenging application for modal parameter estimators is in the airworthiness certification of new aircraft. Manufacturers are required to demonstrate in flight that their aircraft is free from aeroelastic instabilities (such as flutter) throughout the entire flight envelope [21, 22]. During such a flight test, the aircraft is excited and its response measured to verify stabil-
ity, e.g. via the damping ratios of the modes. Often a classical experimental modal analysis (the ground vibration test or GVT) is performed on the aircraft before the flights to validate the mathematical models and to predict the critical speeds. Wright and Cooper [23] provide an extensive overview of the entire process.

In flight flutter testing, a consistent and efficient estimate of the damping ratios is critical to assess the margin to flutter and hence the safety of the aircraft, but the algorithms must also be fast to warn early for a decreasing damping ratio. Therefore a trade-off is needed between accuracy of the estimate and computation time of the algorithm. Our main contribution is the introduction of new frequency-domain modal parameter estimators that are fast, consistent, and polyreference, by combining the strengths of existing estimators.

Classical frequency-domain estimators do not have all these properties at once:

- the polyreference LSCF is fast but not consistent [14],
- generalised-total-least-squares (GTLS) based estimators are consistent [24, 25] but do not use multiple references and are not optimised for computational speed.

We have used the right matrix-matrix description formulation of the LSCF (allowing multiple references) and combined it with the weighted error equation formulation of the GTLS (resulting in consistent estimates). Then we have computationally optimised the algorithm along the same lines as the LSCF, eventually resulting in fast, and consistent estimators using multiple references to separate closely-spaced modes.

We first review the LSCF estimator in Section 2. Then we derive the new polyreference estimators based on the (consistent) generalised total-least-squares strategy (Section 3) and the (quasi-consistent) iterative-quadratic-maximum-likelihood method (Section 4). The performance of these estimators is tested on simulated ground vibration test data in Section 5, and then demonstrated on real-life flight flutter test data (Section 6).

2. The polyreference LSCF estimator

The polyreference LSCF is based on a right matrix-fraction description of the form

$$H(Ω_k, θ) = N(Ω_k, θ)D^{-1}(Ω_k, θ)$$

(1)
where $H(\Omega_k, \theta)$ is the FRF model, with numerator $N(\Omega_k, \theta)$ and denominator $D^{-1}(\Omega_k, \theta)$, $\Omega_k$ is the polynomial base function at frequency line $k = 0, 1, \ldots, F$ (e.g. $\Omega_k = j\omega_k$, $\Omega_k = \exp(j\omega_k T)$, or orthogonal polynomials can be used), and $\theta$ the $(n+1)(N_o+N_i) \times N_i$ parameter matrix containing the coefficients to be estimated. Since it is assumed that the outputs are uncorrelated (which is exactly true for scanning LDV measurements and the reciprocal of a roving hammer test), each output (subscript $o$) can be considered separately:

$$H_o(\Omega_k, \theta) = N_o(\Omega_k, \theta)D^{-1}(\Omega_k, \theta)$$  \hspace{1cm} (2)

The linearised (weighted) least-squares (LS) equation error $\varepsilon_o(\Omega_k, \theta, H_o)$ is given by

$$\varepsilon_o(\Omega_k, \theta, H_o) = W_o(\Omega_k)(N_o(\Omega_k, \theta) - H_o(\omega_k)D(\Omega_k, \theta))$$  \hspace{1cm} (3)

with $H_o(\omega_k) \in \mathbb{C}^{1 \times N_i}$, $(o = 1, \ldots, N_o)$ the $o$th row of the measured frequency response function vector, relating the output $o$ to all $N_i$ inputs, at circular frequency $\omega_k$. $W_o(\Omega_k)$ is an optional diagonal frequency-dependent weighting matrix that can be used to increase the efficiency of the estimates. Proper choices of this weighting will be discussed in more detail in the next sections.

Denoting $\varepsilon_o(\Omega_k)$ by $\varepsilon_{o,k}$ for the sake of compactness, the cost function $\ell(\theta, H)$ can be written as

$$\ell(\theta, H) = \sum_{o=1}^{N_o} \sum_{k=1}^{F} \varepsilon_{o,k}\varepsilon_{o,k}^H = \sum_{o=1}^{N_o} \sum_{k=1}^{F} \text{tr}\left(\varepsilon_{o,k}\varepsilon_{o,k}^H\right)$$  \hspace{1cm} (4)

where $(\cdot)^H$ denotes the Hermitian (complex conjugate) transpose, and $\text{tr}(\cdot)$ is the trace operator. As Eq. (3) is linear in the parameters, it can be rewritten as

$$\varepsilon = J\theta \approx 0$$  \hspace{1cm} (5)

with $\varepsilon = [\varepsilon_1^T, \ldots, \varepsilon_{N_o}^T]^T$ and $\varepsilon_o = [\varepsilon_{o,1}^T, \ldots, \varepsilon_{o,F}^T]^T$, where $(\cdot)^T$ denotes the matrix transpose. $J$ is the Jacobian matrix. If the FRF model in Eq. (1) is restricted to numerator and denominator polynomials with real coefficients (i.e. $\theta$ is considered real), then real equations must be used:

$$\varepsilon_{re} = J_{re}\theta \approx 0$$  \hspace{1cm} (6)
with \( \mathbf{J}_{re} = \begin{bmatrix} \text{Re}(\mathbf{J}) \\ \text{Im}(\mathbf{J}) \end{bmatrix} \), and \( \text{Re}(.) \) and \( \text{Im}(.) \) denote taking the real and imaginary part respectively. If complex coefficients are assumed in Eq. (1), \( \theta \) is complex and Eq. (5) can be used directly. We assume that \( \theta \) is complex in this paper.

Minimising Eq. (4) corresponds to solving

\[
\mathbf{J} \theta = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 & \Upsilon_1 \\ 0 & \Gamma_2 & \cdots & 0 & \Upsilon_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Gamma_{N_o} & \Upsilon_{N_o} \end{bmatrix} \begin{bmatrix} \theta_{N_1} \\ \theta_{N_2} \\ \vdots \\ \theta_{N_{N_o}} \\ \theta_D \end{bmatrix} \approx 0 \tag{7}
\]

where

\[
\theta_{N_o} = \begin{bmatrix} N_{o,n} \\ N_{o,n-1} \\ \vdots \\ N_{o,0} \end{bmatrix} \quad \text{and} \quad \theta_D = \begin{bmatrix} D_n \\ D_{n-1} \\ \vdots \\ D_0 \end{bmatrix} \tag{8}
\]

so that

\[
\mathcal{N}(\Omega_k, \theta) = \sum_{i=0}^{n} N_{o,i} \Omega_k^i \quad \text{and} \quad \mathcal{D}^{-1}(\Omega_k, \theta) = \sum_{i=0}^{n} D_i \Omega_k^i \tag{9}
\]

and with

\[
\Gamma_o = \begin{bmatrix} \Gamma_o(\Omega_1)^T \\ \Gamma_o(\Omega_2)^T \\ \vdots \\ \Gamma_o(\Omega_F)^T \end{bmatrix}^T \tag{10}
\]

\[
\Upsilon_o = \begin{bmatrix} \Upsilon_o(\Omega_1)^T \\ \Upsilon_o(\Omega_2)^T \\ \vdots \\ \Upsilon_o(\Omega_F)^T \end{bmatrix}^T \tag{11}
\]

where

\[
\Gamma_o(\Omega_k) = W_o(\omega_k)[\Omega_k^n, \Omega_k^{n-1}, \ldots, \Omega_k^0] \tag{12}
\]

\[
\Upsilon_o(\Omega_k) = -\Gamma_o(\Omega_k) \otimes \mathbf{H}_o(\omega_k) \tag{13}
\]

\( \otimes \) is the Kronecker product [26]. Note that \( \mathbf{J} \) is independent of the parameters to be estimated. \( \mathbf{J} \) has size \( N_o F \times (n+1)(N_o+N_i) \) and \( \theta \) \((n+1)(N_o+N_i) \times N_i\).

Formulating the normal equations gives a more compact expression:

\[
\mathbf{J}^H \mathbf{J} \theta = \begin{bmatrix} \mathbf{R}_1 & \cdots & 0 & \mathbf{S}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \mathbf{R}_{N_o} & \mathbf{S}_{N_o} \\ \mathbf{S}_1^H & \cdots & \mathbf{S}_{N_o}^H & \sum_{o=1}^{N_o} \mathbf{T}_o \end{bmatrix} \theta \approx 0 \tag{14}
\]
with $R_o = \Gamma_o^H \Gamma_o$, $S_o = \Gamma_o^H \Upsilon_o$ and $T_o = \Upsilon_o^H \Upsilon_o$. The $R_o$, $S_o$ and $T_o$ submatrices have a Toeplitz structure enabling a fast construction [14]

$$
[R_o]_{rs} = \sum_{k=0}^{F-1} |W_o(\omega_k)|^2 \Omega_k^{s-r}
$$

$$
[S_o]_{rl} = -\sum_{k=0}^{F-1} |W_o(\omega_k)|^2 \Upsilon_o(\omega_k) \Omega_k^{s-r}
$$

$$
[T_o]_{kl} = \sum_{k=0}^{F-1} |W_o(\omega_k)|^2 \Upsilon_o^H(\omega_k) \Upsilon_o(\omega_k) \Omega_k^{s-r}
$$

with $k = [(r-1)N_i+1 : rN_i]$, $l = [(s-1)N_i+1 : sN_i]$ for both $r, s = 1, 2, \cdots, n+1$.

Again, if $\theta$ is constrained to real-valued coefficients, only the real part in Eq. (14) must be taken into account, which gives

$$
\text{Re} (J^H J) \theta \approx 0
$$

Note however that the number of identified modes is equal to $nN_i$ for the complex case and $\frac{nN_i}{2}$ for a real coefficients matrix $\theta$.

Elimination of the numerator coefficients $\theta_{N_o}$ from Eq. (14) by substitution of

$$
\theta_{N_o} = -R_o^{-1} S_o \theta_D
$$

results in the reduced normal equations

$$
\left[ \sum_{o=1}^{N_o} (T_o - S_o^H R_o^{-1} S_o) \right] \theta_D = M \theta_D = 0
$$

with $\theta_D$ the denominator coefficients, and $M$ a square $(n+1)N_i \times (n+1)N_i$ matrix which is much smaller than the original normal equations matrix in Eq. (14). The least-squares (LS) solution of Eq. (18) is found by fixing one of the denominator coefficients, e.g. the one corresponding to the highest order coefficient, to the identity matrix $I_{N_i}$

$$
\theta_D = \left[ I_{N_i} - [M(N_i+1:(n+1)N_i, N_i+1:(n+1)N_i)]^{-1}[M(N_i+1:(n+1)N_i, 1:N_i)] \right]^{-1}
$$

(19)
The consistency of an estimator is ensured when the limit of the expected value of the cost function is minimal in the true parameter $\theta$, under some specific assumptions on the cost function [20, Theorem 15.15]. Practically speaking, if the cost function is consistent, then the minimiser of this cost function (the estimate) is also consistent. This condition is easily verified by calculating the expected value of the cost function, and evaluating whether or not this function is minimal in the true value $\theta$. The description of this method to verify consistency is based on Pintelon and Schoukens [20, p. 190].

Since we limit ourselves to equation errors of the form as in Eq. (3), which are linear in the measurements, the error equation can be split up as

$$\ell(\theta, H) = \text{tr} \left( \left( \varepsilon(H, \theta)^H \varepsilon(H, \theta) \right) + \left( \varepsilon(\Delta H, \theta)^H \varepsilon(\Delta H, \theta) \right) \right)$$

$$+ 2 \text{herm} \left( \varepsilon(H, \theta)^H \varepsilon(\Delta H, \theta) \right)$$

$$= \ell(\theta, H) + \ell(\theta, \Delta H) + \text{tr} \left( 2 \text{herm} \left( \varepsilon(H, \theta)^H \varepsilon(\Delta H, \theta) \right) \right)$$

(20)

where we used that $\varepsilon(H, \theta) = \varepsilon(H, \theta) + \varepsilon(\Delta H, \theta)$, with $H$ the true FRF and $\Delta H$ the noise on the FRF, and $\text{herm}(A) = (A + A^H)/2$. Since we also assume that $H$ and $\Delta H$ are independent, the expected value (denoted as $E\{., \}$) of the cost function is found as

$$E\{\ell(\theta, H)\} = E\{\ell(\theta, H)\} + E\{\ell(\theta, \Delta H)\}$$

(21)

The consistency can now be assessed by evaluating Eq. (21) in $\theta$ to verify if the cost function is minimal:

$$E\{\ell(\theta, H)\} = E\{\ell(\theta, H)\} + E\{\ell(\theta, \Delta H)\}$$

(22)

The first term of the right hand side is zero since the true model parameters $\theta$ exactly represent the true noiseless measurements $H$, assuming no model errors. This last assumption requires that a true model exists, which is reasonable in general [20]. The second term of Eq. (22) eventually reveals (in)consistency: if this term is independent of $\theta$, then Eq. (22) is always minimal in $\theta$ and the estimator is consistent. If this term is a function of $\theta$, then it is possible that other minimisers of Eq. (21) exist, and the estimator is inconsistent.
Elaborating the second term of Eq. (21) for the LSCF, gives

\[
E\{\ell(\theta, H)\} = E\{\ell(\theta, H)\} + E\left\{\text{tr}\left(\varepsilon(\Delta H, \theta)^H \varepsilon(\Delta H, \theta)\right)\right\}
\]

\[
= E\{\ell(\theta, H)\} + \sum_{k=1}^{F} \sum_{o=1}^{N_o} \text{tr}\left(D_k^H(\Omega_k, \theta)\text{Cov}(H_{o,k})D_k(\Omega_k, \theta)\right)
\]

(23)

with

\[
\text{Cov}(H_{o,k}) = E\{\Delta H_{o,k}^H \Delta H_{o,k}\}
\]

(24)

The polyreference LSCF is inconsistent as Eq. (23) is dependent on \( \theta \) and thus not minimal in the true parameter \( \theta \). It is consistent in the noiseless case (\( \text{Cov}(H_{o,k}) = 0 \)), or in the noisy case when the denominator polynomial is a constant (\( D_k(\Omega_k) = aI_N \)), thus for a system containing no poles, only zeros.

3. Polyreference WGTLS estimators

The estimates can also be found in a mixed least-squares total-least-squares (LS-TLS) way, as introduced in [27, p. 92] and applied by Verboven et al. [13] to the (common-denominator) frequency-domain modal parameter estimators. A TLS estimator constrains the norm-1 of the full parameter \( \theta \), while a mixed LS-TLS algorithm imposes the norm-1 constraint only on the denominator parameters \( \theta_D \). For our polyreference mixed LS-TLS, the constraint is given by

\[
\theta_D^H \theta_D = I_N
\]

The estimates are found as the \( N_i \) eigenvectors \( V_r \) corresponding to the \( N_i \) smallest eigenvalues \( \lambda_r \) of the eigenvalue decomposition of Eq. (18)

\[
MV_r = \lambda_r V_r
\]

(25)

Before analysing the consistency of the polyreference mixed LS-TLS estimator, we first derive mixed Least Squares-Weighted Generalised Total Least Squares (mixed LS-WGTLS) estimators. The TLS is a special version of the WGTLS (we drop the ‘mixed LS-’ notation in the remainder of this paper, since all our WGTLS estimators are mixed LS-WGTLS).

The frequency-domain WGTLS estimator has been derived in [25] for a common-denominator and left matrix-fraction description. In the remainder of this section, we derive polyreference WGTLS estimators based on a
right matrix-fraction model, and we prove the consistency. We start from
the equation error in the formulation of Eq. (5), but we have removed the
weighting from $J$, i.e.
\[
\varepsilon(H, \theta) = W J \theta \approx 0
\]  
(26)

where
\[
\{J\}_{k,o} = \frac{\partial \varepsilon_o(\Omega_k)}{\partial \theta^T} = \frac{\partial}{\partial \theta^T} (N_o(\Omega_k, \theta) - H_o(\omega_k) D(\Omega_k, \theta))
\]  
(27)

Our derivation for the RMFD follows the same lines as [25, Sec. 3] for the
LMFD. The WGTLS estimate for Eq. (26) is given by [27, 25]
\[
\arg\min_{\tilde{J}, \theta} \|W(J - \tilde{J})C^{-1}\|_F^2
\]  
subject to $\tilde{J} \theta = 0$ and $\theta^H \theta = I_{N_i}$, where $\tilde{J}$ is a modified $J$ so that $\tilde{J}\theta$ is
exactly zero (the first constraint), and with the norm-1 constraint on the
estimates (the second constraint). The Frobenius norm $\|\cdot\|_F$ of a matrix is
defined as
\[
\|A\|_F^2 = \sum_p \sum_q |A_{pq}|^2 = tr(A^H A)
\]  
(29)

The right weighting matrix $C$ is the square root of the column covariance
matrix of $WJ$, given by
\[
C^H C = E\{\Delta J^H W^H W \Delta J\}
\]  
(30)

with $\Delta J = (J - E\{J\})$ the noise contribution to the Jacobian matrix caused
by the uncertainty on the FRF data. The matrix $C$ is singular when one
or more DFT spectra are noise-free, or when some of the noise sources are
totally correlated [25]. The resulting numerical problems can be avoided by
using the SVD, as we discuss in Section 3.4.

The equivalent cost function is found by eliminating $\tilde{J}$ from Eq. (28), as
elaborated in Appendix A. The cost function is then reformulated as
\[
\ell(H, \theta) = tr \left( [\theta^H C^H C \theta]^{-1} [\theta^H J^H W^H W J \theta] \right)
\]  
subject to $\theta^H \theta = I_{N_i}$. We derive the different TLS, GTLS, and WGTLS
from Eq. (31) in the next paragraphs.
3.1. Total Least Squares

By setting \( W = I_{N_o F} \) and \( C = I_{(n+1)(N_o+N_i)} \) in Eq. (31), the Total Least Squares (TLS) cost function is found to be

\[
\ell(\theta, H) = \text{tr} \left( [\theta^H \theta]^{-1} [\theta^H J^H J \theta] \right)
\]
\[
= \sum_{k=1}^{F} \sum_{o=1}^{N_o} \text{tr} \left( [\theta^H \theta]^{-1} [\epsilon_o^H(\Omega_k, \theta) \epsilon_o(\Omega_k, \theta)] \right)
\]  

(32)

We now apply the tool described in Section 2 to determine whether or not the TLS estimates are consistent. The expected value of the cost function Eq. (32) is found as

\[
E\{\ell(\theta, H)\} = E\{\ell(\theta, H)\} + \sum_{k=1}^{F} \sum_{o=1}^{N_o} \text{tr} \left( [\theta^H \theta]^{-1} (D^H(\Omega_k, \theta) \text{Cov}(H_{o,k}) D(\Omega_k, \theta)) \right)
\]  

(33)

As the expected value of the cost function is clearly dependent on \( \theta \), it is not minimal in the true parameter value \( \theta \). We conclude that the TLS is not consistent in general. It is consistent if \( \text{Cov}(H_{o,k}) \) is proportional to the unity matrix, as we will show in Eq. (45). If the polynomial basis function is formulated in the continuous Laplace-domain \( (\Omega_k = j\omega_k) \), the TLS estimator suffers from the overemphasising of the high frequency errors due to the lack of a frequency dependent weighting [24]. This can be alleviated by using the discrete-time Z-domain.

3.2. Generalised Total Least Squares

The Generalised Total Least Squares (GTLS) estimator can be found from Eq. (31) by setting \( W = I_{N_o F} \). Taking into account Eq. (30), the cost function becomes:

\[
\ell(\theta, H) = \sum_{k=1}^{F} \sum_{o=1}^{N_o} \text{tr} \left( \left[ \sum_{l=1}^{F} \sum_{p=1}^{N_o} (D^H(\Omega_l, \theta) \text{Cov}(H_{p,l}) D(\Omega_l, \theta)) \right]^{-1} \times [\epsilon_o^H(\Omega_k, \theta) \epsilon_o(\Omega_k, \theta)] \right)
\]  

(34)
\[ \theta^H C^H C \theta = E \{ \theta^H J^H J \theta \} \]
\[ = \sum_{l=1}^{F} \sum_{p=1}^{N_o} (D^H(\Omega_l, \theta) \text{Cov}(H_{p,l}) D(\Omega_l, \theta)) \]  
\[ (35) \]

The expected value of the cost function Eq. (34) is then found as
\[ E\{ \ell(\theta, H) \} = E\{ \ell(\theta, H) \} + N_i \]  
\[ (36) \]

which is minimal in the true parameters \( \theta \); the GTLS estimates are thus consistent. As every frequency is weighted with the same factor, the GTLS suffers from the amplification of high frequency errors if the Laplace domain is used.

### 3.3. Approximate Bootstrapped Total Least Squares

Comparing the TLS and the GTLS reveals that the \textit{a priori} knowledge of \( C^{-1} \) leads to consistent estimates. The use of a left weighting matrix \( W \) clearly does not affect the consistency. However, this weighting does influence the \textit{efficiency} of the estimator. The optimal weighting is the Maximum Likelihood weighting:
\[ W^{-2}_{\text{ML},o}(\Omega_k, \theta) = D^H(\Omega_k, \theta) \text{Cov}(H_{o,k}) D(\Omega_k, \theta) \]  
\[ (37) \]

There are however two problems if one wants to apply this weighting to the WGTLS:

- the weighting is a function of \( \theta \),
- the weighting is a \( N_i \times N_i \) matrix per output \( o \) and frequency \( k \) (scalar if \( N_i = 1 \)).

Both problems cannot be solved without sacrificing speed of the algorithm.

The first problem is fundamental. The weighting cannot be calculated as \( \theta \) is unknown. The MLE estimator circumvents this by minimising the cost function using a non-linear optimisation algorithm (such as Gauss-Newton or Levenberg-Marquardt, see e.g. 28). Non-linear solvers need good starting values. An initial estimate of \( \theta \) can be obtained by means of the LS, TLS, or GTLS estimator. These parameters can then be used to calculate an estimate.
of the MLE weighting in Eq. (37). This approach could be repeated, where
the new parameter estimates are used to calculate a better approximation of
the MLE weighting. The resulting iterative WGTLS estimator is called the
Bootstrapped Total Least Squares (BTLS) estimator [24].

The second problem is related to the left multiplication of the Jacobian
matrix with frequency dependent weighting matrices. Indeed, by left multi-
plying $J$ with a diagonal $W$, each row of $J$ is weighted in the same way. This
is only possible if $W$ in Eq. (28) is a diagonal matrix with entries $W_o(\Omega_k)$,
thus a scalar weighting per output $o$ and per frequency $k$.

A solution used by Pintelon et al. [25] is to make a scalar appr oximation
of the MLE weighting by using the trace of the weighting matrix Eq. (37),
using the estimates $\theta(i-1)$ of the previous iteration step:

$$W_o^{-2}(\Omega_k, \theta(i-1)) = tr \left( D^H(\Omega_k, \theta(i-1)) \text{Cov}(H_o,k) D(\Omega_k, \theta(i-1)) \right)$$
$$= tr W_{ML,o}^{-2}(\Omega_k, \theta(i-1)) \quad (38)$$

We define this approach the Bootstrapped Total Least Squares (BTLS) es-
timator, analogous to the common-denominator algorithm. H owever, as the
weighting is only an approximation of the MLE weighting, the estimator will
not have the same efficiency as a true BTLS estimator. The efficiency will
still be improved compared with the unweighted GTLS and TLS. Defining
$(i)\epsilon_o,k = \epsilon_o(\Omega_k, \theta(i))$ and $(i)\epsilon_{o,k} = W_{ML,o}(\Omega_k, \theta(i))$, the $(i)^{th}$ estimate $\theta(i)$ is
the minimiser of the cost function $\ell(H, \theta(i))$, given by

$$\ell(\theta(i)) = \sum_{k=1}^{F} \sum_{o=1}^{N_o} tr \left( \sum_{l=1}^{F} \sum_{p=1}^{N_p} \left[ \left( \frac{(i)W_{p,l}^{-2}}{tr[(i-1)\epsilon_{o,k}W_{p,l}^{-2}]} \right)^{-1} \right] \epsilon_{o,k} \epsilon_{o,k}^H \right) \quad (39)$$

The first iteration of the polyreference approximate BTLS is consistent, as
the expected value of the cost function is minimal in the true parameters:

$$E\{\ell(\theta, H)\} = E\{\ell(\theta, H)\} + N_i \quad (40)$$

Moreover, Pintelon and Schoukens [20, p. 219] have proven that the esti-
mates are consistent in each iteration step. From a theoretical point of view,
this is an advantage compared with the MLE, since the MLE estimates are
only consistent when the optimisation algorithm converges. However, the
uncertainty on the MLE estimates is lower, since the MLE is an efficient
estimator.
3.4. Practical calculation of the WGTLS estimates

It is shown in Pintelon and Schoukens [20, Appendix 7.1] that the solution of Eq. (31) can be calculated by means of the Generalised Eigenvalue Decomposition (GED) of

$$J^H W^H W J V_r = \lambda_r C^H C V_r$$  \hspace{1cm} (41)

where the solution $\theta$ is given by the $N_i$ eigenvectors $V_r$ corresponding to the $N_i$ smallest eigenvalues $\lambda_r$. A numerically more stable calculation is the Generalised Singular Value Decomposition (GSVD) of $(WJ, C)$ [20, p. 209]. The estimated parameters are then the $N_i$ right singular vectors corresponding to the $N_i$ smallest singular values. This calculation is possible even when $C$ is singular.

The covariance matrix $C^H C$ has the same structure as $J^H J$

$$C^H C = \begin{bmatrix} C_{R_1} & \cdots & 0 & C_{S_1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & C_{R_{No}} & C_{S_{No}} \\ C_{S_1}^H & \cdots & C_{S_{No}}^H & \sum_{o=1}^{No} C_{T_o} \end{bmatrix}$$ \hspace{1cm} (42)

Since the submatrices $R_o$ of the Jacobian matrix are free from noise (see Eq. (12)), the matrices $C_{R_o}$ and $C_{S_o}$ are zero. The WGTLS normal equations can be reformulated as a reduced set of equations, resulting in a very compact formulation [see 13, for the common denominator]:

$$M V_r = \lambda_r C_T V_r$$ \hspace{1cm} (43)

with

$$M = \sum_{o=1}^{No} (T_o - S_o^H R_o^{-1} S_o)$$

$$C_T = \sum_{o=1}^{No} C_{T_o}$$ \hspace{1cm} (44)

and subject to the constraint $\theta_D^H \theta_D = I_{N_i}$.

If the covariance matrix of $C^H C$ in Eq. (41) or $C_T$ in Eq. (43) is proportional to the unity matrix,

$$C^H C = \sigma^2 I_{(n+1)(N_o+N_i)}$$

$$C_T = \sigma^2 I_{(n+1)N_i}$$ \hspace{1cm} (45)

with $\sigma \in \mathbb{R}$, $\sigma \neq 0$, then the GTLS equals the TLS estimator. In this special case, the ‘normal’ TLS estimator is also consistent.
4. Approximate Iterative Quadratic Maximum Likelihood

The same scalar frequency weighting (Eq. 38) used for the approximate BTLS estimator can be applied to produce an iterative weighted LS or TLS estimator. This approach is called the Iterative Quadratic Maximum Likelihood (IQML) estimator, in analogy with the common-denominator algorithm. However, as for the polyreference BTLS, the use of the trace of the MLE weighting will only result in an approximate IQML. The cost function is given by, for the weighted LS:

$$\ell(\theta(i), H) = \sum_{k=1}^{F} \sum_{o=1}^{N_o} \left( \frac{\text{tr} \left( \varepsilon_o^H(i) \varepsilon_o \right)}{\text{tr} (i-1) W_{o,k}^{-2}} \right)$$

and, for the weighted TLS:

$$\ell(\theta(i), H) = \sum_{k=1}^{F} \sum_{o=1}^{N_o} \text{tr} \left( \left[ \theta^H \theta \right]^{-1} \left[ \varepsilon_o^H(\Omega_k, \theta)(i) \varepsilon_o(\Omega_k, \theta) \right] \right)$$

It is not possible to verify consistency for $\theta(\infty)$ of the IQML estimator in general [20, p. 218 and 202]. It is however possible to evaluate the first iteration estimate $\hat{\theta}(1)$, under the assumption that the starting values $\theta(0)$ are obtained using a LS or TLS, optionally with a deterministic weighting. The asymptotic properties are equal to the LS and TLS estimators respectively, as is found from the expected values of the cost function, for the LS version:

$$E\{\ell(\theta(1), H)\} = E\{\ell(\theta(1), H)\} + \sum_{k=1}^{F} \sum_{o=1}^{N_o} \frac{\text{tr} \left( \varepsilon_o(i) W_o^{-2} \right)}{\text{tr} (0) W_o^{-2}}$$

and for the TLS-based estimator:

$$E\{\ell(\theta(1), H)\} = E\{\ell(\theta(1), H)\} + \sum_{k=1}^{F} \sum_{o=1}^{N_o} \frac{\text{tr} \left( \theta_{(1)^T} \theta_{(1)} \varepsilon_o(i) W_o \right)}{\text{tr} (0) W_o^{-2}}$$

Since the expected value is dependent on $\theta$, the IQML is not consistent. However, if the IQML converges ($\theta(i) = \theta(i-1)$), then the estimates will be consistent. Pintelon et al. [24] state that for sufficiently high signal-to-noise ratios and sufficiently low modelling errors, the IQML and the ML estimates will coincide. But as only an approximation of the MLE weighting is used
for the polyreference IQML, it is expected that the accuracy of the estimates will improve compared with the LS and TLS algorithms, but not as good as a ‘full’ IQML (e.g. a single-reference common-denominator model based IQML).

5. Simulation based on a ground vibration test

The first part of airworthiness certification of a newly developed aircraft is the ground vibration test (GVT), which is used to validate and update the finite element (FE) model [23]. We have used GVT measurements on a small, piston-engined aeroplane, the Mission M212 from Lambert Aircraft Engineering. Figure 1\(^1\) shows a picture (left) and a test model representation of the first bending mode of the aircraft (right). The GVT was performed by LMS International.

Figure 1 around here.

The data set consists of two burst random input signals and 278 acceleration measurements. The aircraft was excited at both left and right wing tips, in the Z-direction. The FRFs were constructed using the auto-power and cross-power spectral densities using the \(H_1\)-estimator. Since one of our goals is to compare the bias on the different estimators, we need to know the true values of the estimates. We therefore chose not to work directly on the measurements, but rather on a simulated (controlled) data set based on the GVT measurements. To this end, we made an initial estimate using the polyreference LSCF estimator. We then selected the seven most dominant modes within a limited band of 171 equally distributed frequencies to construct a synthesised FRF. This FRF is our exact or ‘true’ FRF, and we contaminate it with additive noise to mimic measurements and perform Monte Carlo simulations.

We added both a constant (white) noise term and a flicker (pink) noise term [29] to approximate the low-frequency contamination by sources such as electronic noise, or e.g. in-flight turbulence. Figure 2 shows the exact and sample FRF for one output and one input, and the noise level (SNRs between 40 and 5 dB in the peaks). The Monte Carlo simulations have been

\(^1\)Picture copyrighted by James Grimstead. The test model was provided by LMS International.
performed with 500 runs. For each set the poles have been calculated using the polyreference TLS, GTLS, BTLS, IQML, all with norm-1 constraint ($\Theta_D^H \Theta_D = I_N$), and the polyreference LSCF and IQML with parameter constraint ($D_n = I_N$). The iterative estimators BTLS and IQML are set to two iterations. This two-step approach consists thus of an initial estimate (the starting value, i.e. a GTLS or a LS/TLS estimate respectively) and one update with the weighting calculated from the starting value. More iterations are possible, but normally the two-step approach suffices to obtain an important improvement.

The systematic error is assessed using the bias $b$ and the (Root) Mean Squared Error (MSE) of the mean of the pole estimates $\hat{p}$. These are given by

$$b = E\{\hat{p}\} - p_e$$  \hspace{1cm} (50)

$$\text{RMSE} = \sqrt{b^2 + \sigma^2}$$  \hspace{1cm} (51)

with $p_e$ the exact poles and $\sigma^2$ the variance of the pole estimates (unit: rad/s). The absolute values of the bias and RMSE are shown in Tables 1 and 2 respectively.

It follows that the TLS improves the LS estimates by a factor between 1.1 and 10 in bias, while the GTLS is still better: the bias is about one order of magnitude smaller compared with the LS estimator. The GTLS is the one-step approach with the lowest bias. The iterative BTLS uses the GTLS as starting values but still maintains consistency. Comparing the bias of GTLS and BTLS it follows that, even after only one additional iteration, the bias on the BTLS estimates is further reduced with a factor of between 2 and more than 10, depending on the mode considered. The two-step BTLS offers a consistent estimate with only a limited increase in computing time (see also next section). The IQML with norm-1 constraint does not perform quite as well, with a bias comparable to the bias on the TLS estimates, but the IQML with parameter constraint always has a lower RMSE than the LSCF, up to almost an order of magnitude.
6. Demonstration on flight flutter test data

The flutter test was performed on a commercial aircraft equipped with fly-by-wire control. This primary flight control system was also used as excitation device: the electric force signal was superposed on the (auto) pilot’s control signal and fed to the ailerons. The excitation signal was a sine-sweep of 120 s that was sent to both ailerons, but with a phase lag of $\pi$ radians, resulting in an anti-symmetrical excitation. Multiple accelerometers were used to capture the response of the aircraft, but we limited our analysis to four outputs and the two inputs. Figure 3 shows the force signal for the port aileron, and one response at the port wing tip, and the corresponding FRF and the sample noise variance. (Note that the frequency axis has been shifted for confidentiality reasons as required by the aircraft’s manufacturer.) The FRF and its variance are estimated using the $H_1$ approach with 5 averages and 50% overlap. Then a frequency band of interest is chosen with 192 frequency lines.

We curve-fitted the data with a model order of 32, and we again limited the iterative estimators to two iterations. The computation times for this test (obtained on 2.40 GHz processor with 6 GB RAM and Ubuntu OS) are shown in Table 3. The increases in computation time compared with the LSCF vary between 0.5 and 1.1. The curve-fitting results are shown in Figure 4.

The figures are positioned so that the bottom row is always the weighted and iterative version of the top row. The left figures are the estimators with the parameter constraint (LSCF and IQML): they obtain a similar fit. For the center figures (TLS and IQML with norm-1 constraint), the improvement after one iteration is important: the ‘spikes’ of the TLS are removed and a mode at 7.3 Hz is included. On the right, the BTLS improves the GTLS estimate at the higher frequencies and is able to estimate a highly damped mode at 8 Hz (as LSCF and IQML estimators).

Comparing the bottom with top row, it can be seen that the anti-resonance
at 6.3 Hz is not followed by the iterative, weighted estimators. This can be
attributed to the use of the noise variance as a weighting in the estimation:
less weight is given to the noisier parts of the FRF, and the noise level in
this particular region is higher than the FRF (see Figure 3, right). This
also explains why the ‘bottom’ estimators do not find a mode at 5.3 Hz: the
noise level is of the same order of magnitude as the FRF itself; i.e. the data
is not reliable. But the ‘blind’ LSCF and TLS estimators do identify a mode
there, and also the GTLS needs a BTLS iteration to disregard what the noise
variance predicts is a disturbance.

7. Conclusions

In this paper we have derived the polyreference implementation of exist-
ing WGTLs estimators. More specifically, we have used the right matrix-
fraction description model and the least-squares complex frequency-domain
algorithm to construct fast, polyreference versions of the GTLS, BTLS, and
IQML estimators. These estimators are consistent: they will yield the correct
parameters if an infinite amount of data were available. In this way, they
form a compromise between the fast but inconsistent polyreference LSCF
estimator, and the slower but consistent and efficient ML estimator. There-
fore, these estimators are suited to handle flight flutter test data, where
the calculation time should be limited, but where also the accuracy of the
(damping ratio) estimate is important for the safety of the flight. Moreover,
since the estimators are polyreference, they perform better for the estima-
tion of closely-spaced poles compared with single-reference estimators. This
is again needed for flutter testing due to the large structures and thus nu-
merous modes in a low frequency band.

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ders (IWT); The Concerted Research Action ‘OPTIMech’ of the Flemish
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(VUB) and the department of Industrial Sciences & Technology of the Eras-
mushogeschool Brussel (EhB) are gratefully acknowledged.
Appendix A. Derivation of the WGTLS equivalent cost function

This proof follows the same lines as Pintelon et al. [25, Appendix] and Pintelon and Schoukens [20, Appendix 7.H]. The proof will show that both Eqs. (28) and (31) have the same stationary points. Reformulating Eq. (28) with Lagrangian multipliers to import the constraint $\tilde{J}\theta = 0$, gives

$$\ell(\theta, H) = tr \left( W(J - \tilde{J})[C^H C]^{-1}(J - \tilde{J})^H W^H \right) + tr \left( \Lambda^H \tilde{J}\theta \right)$$  \hspace{1cm} (A.1)

subject to $\theta^H \theta = I_{N_i}$, with $\Lambda$ the $N_o N \times N_i$ matrix of Lagrange multipliers.

This cost function must be stationary in its minima with respect to $\theta, \tilde{J}$ and $\Lambda$. The derivative of Eq. (A.1) with respect to $\tilde{J}$ is the derivative of a real function of a complex argument and its complex conjugate, and thus not possible in a straightforward way. We therefore use the symbolic derivative [see e.g. 20, p. 423], where we derive with respect to the real and imaginary parts of $\tilde{J}$ respectively. Using Appendix B, this derivative is found as

$$-2[W^H W][J - \tilde{J}][C^H C]^{-1} + \Lambda \theta^H = 0$$ \hspace{1cm} (A.2)

As $\tilde{J}\theta = 0$, right multiplying with $\theta$ yields

$$\Lambda = 2[W^H W][J - \tilde{J}]\theta[\theta^H C^H C\theta]^{-1}$$ \hspace{1cm} (A.3)

Substituting $\Lambda$ in Eq. (A.2) gives

$$J - \tilde{J} = J\theta[\theta^H C^H C\theta]^{-1}\theta^H C^H C$$ \hspace{1cm} (A.4)

Back substituting $J - \tilde{J}$ in Eq. (A.1) gives

$$\arg \min_{\theta} tr (\theta^H C^H C\theta)^{-1} \theta^H J^H W^H W J\theta$$ \hspace{1cm} (A.5)

subject to $\theta^H \theta = I_{N_i}$.

Appendix B. Derivative of a matrix

Given two matrices $A(m \times n)$ and $B(p \times q)$, the matrix derivative has dimension $(mp \times nq)$, and is defined as

$$\frac{\partial A}{\partial B} = \begin{bmatrix} \frac{\partial A}{\partial b_{11}} & \frac{\partial A}{\partial b_{12}} & \cdots & \frac{\partial A}{\partial b_{1q}} \\ \frac{\partial A}{\partial b_{21}} & \frac{\partial A}{\partial b_{22}} & \cdots & \frac{\partial A}{\partial b_{2q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A}{\partial b_{p1}} & \frac{\partial A}{\partial b_{p2}} & \cdots & \frac{\partial A}{\partial b_{pq}} \end{bmatrix}$$  \hspace{1cm} (B.1)
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Table 1: Comparison of the absolute value of the bias based on simulated Ground Vibration Test data.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td>LSCF¹</td>
<td>0.347</td>
<td>2.188</td>
<td>1.031</td>
<td>0.293</td>
<td>0.154</td>
<td>0.086</td>
<td>0.243</td>
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<tr>
<td>TLS</td>
<td>0.148</td>
<td>1.752</td>
<td>0.258</td>
<td>0.266</td>
<td>0.015</td>
<td>0.014</td>
<td>0.104</td>
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<tr>
<td>GTLS</td>
<td>0.122</td>
<td>0.073</td>
<td>0.040</td>
<td>0.045</td>
<td>0.054</td>
<td>0.005</td>
<td>0.039</td>
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<tr>
<td>BTLS</td>
<td>0.007</td>
<td>0.049</td>
<td>0.005</td>
<td>0.001</td>
<td>0.004</td>
<td>0.003</td>
<td>0.007</td>
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<tr>
<td>IQML¹</td>
<td>0.241</td>
<td>1.694</td>
<td>0.190</td>
<td>0.038</td>
<td>0.096</td>
<td>0.047</td>
<td>0.177</td>
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<tr>
<td>IQML</td>
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<td>0.098</td>
<td>0.015</td>
<td>0.028</td>
<td>0.117</td>
<td>0.365</td>
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(1): \( D_n = I_{N_i} \)
Table 2: Comparison of the Root Mean Squared Error based on simulated Ground Vibration Test data.

<table>
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<th>Estimator</th>
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<th>4</th>
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<td>0.293</td>
<td>0.154</td>
<td>0.087</td>
<td>0.243</td>
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<tr>
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<td>2.301</td>
<td>0.384</td>
<td>0.579</td>
<td>0.024</td>
<td>0.025</td>
<td>0.138</td>
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<tr>
<td>GTLS</td>
<td>0.202</td>
<td>0.333</td>
<td>0.378</td>
<td>0.151</td>
<td>0.179</td>
<td>0.026</td>
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<tr>
<td>BTLS</td>
<td>0.070</td>
<td>0.193</td>
<td>0.076</td>
<td>0.032</td>
<td>0.050</td>
<td>0.073</td>
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<tr>
<td>IQML(^1)</td>
<td>0.241</td>
<td>1.694</td>
<td>0.190</td>
<td>0.038</td>
<td>0.096</td>
<td>0.048</td>
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<tr>
<td>IQML</td>
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<td>0.017</td>
<td>0.029</td>
<td>0.150</td>
<td>0.482</td>
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\(^1\): \(D_n = I_{N_i}\)
Table 3: Sample run times for the different algorithms. Run time per iteration for BTLS and IQML estimators.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>LSCF$^1$</th>
<th>TLS</th>
<th>GTLS</th>
<th>BTLS</th>
<th>IQML$^1$</th>
<th>IQML</th>
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<tbody>
<tr>
<td>Time (s)</td>
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<td>0.2068</td>
<td>0.3301</td>
<td>0.4390</td>
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<td>Relative</td>
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<td>1.00</td>
<td>1.60</td>
<td>2.12</td>
<td>1.64</td>
<td>1.53</td>
</tr>
</tbody>
</table>

($)^1$: $D_n = I_{N_i}$
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4  Sample FRF (red line with dots) and curve-fit (solid black line) of the flutter data. Clockwise from top left: LSCF, TLS, GTLS, BTLS, IQML with norm-1 constraint, IQML with parameter constraint.  33
Figure 1: Picture and test model of the first bending mode of the aircraft.
Figure 2: A typical FRF for the ground vibration test data: exact FRF (black solid line), noise disturbed sample FRF (red line with dots), noise variance (blue dashed line).
Figure 3: A typical time series (left) and FRF (right) for the flight flutter test data. Left: force (blue) and response (green) signals. Right: FRF (red line with dots) and sample noise variance (blue dashed line).
Figure 4: Sample FRF (red line with dots) and curve-fit (solid black line) of the flutter data. Clockwise from top left: LSCF, TLS, GTLS, BTLS, IQML with norm-1 constraint, IQML with parameter constraint.