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Small Weight Codewords of Projective Geometric Codes

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Abstract

We investigate small weight codewords of the p-ary linear code $\mathcal{C}_{j,k}(n,q)$ generated by the incidence matrix of k-spaces and j-spaces of $\mathrm{PG}(n,q)$ and its dual, with q a prime power and $0 \leqslant j < k < n$. Firstly, we prove that all codewords of $\mathcal{C}_{j,k}(n,q)$ up to weight $\left(3 - \mathcal{O}\left(\frac{1}{q}\right)\right) \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q$ are linear combinations of at most two k-spaces (i.e. two rows of the incidence matrix). As for the dual code $\mathcal{C}_{j,k}(n,q)^\perp$, we manage to reduce both problems of determining its minimum weight (1) and characterising its minimum weight codewords (2) to the case $\mathcal{C}_{0,1}(n,q)^\perp$. This implies the solution to both problem (1) and (2) if q is prime and the solution to problem (1) if q is even.

Keywords: Linear codes, Projective spaces, Small weight codewords.
Mathematics Subject Classification: 05B25, 94B05.

1 Introduction

To keep things clear and compact, we will postpone introducing the necessary preliminaries; see Section 3 for an overview of all notations and known results used throughout this article.
A main research topic in coding theory is finding the minimum weight of certain linear codes and characterising its minimum weight codewords (or, more generally, codewords of a relatively small weight). This article investigates small weight codewords of $\mathcal{C}_{j,k}(n,q)$ and $\mathcal{C}_{j,k}(n,q)^\perp$, which are the p-ary linear codes generated by the incidence matrix of k-spaces and j-spaces of $\mathrm{PG}(n,q)$ and its dual, respectively.

Some important characterisations are already known. Szőnyi and Weiner [SW18] characterised all codewords of $\mathcal{C}_{0,1}(2,q)$ up to a certain weight if q is sufficiently large. If $q = p^h$, with p prime, then they characterised codewords up to weight approximately $q\sqrt{q}$ in case $h > 2$, up to weight approximately $\frac{1}{2}q\sqrt{q}$ if $h = 2$, and up to weight $4q - 22$ if $h = 1$.
Using these results, all codewords of $\mathcal{C}_{0,k}(k+1,q)$ up to weight $(3 - \mathcal{O}(\frac{1}{q}))q^k$ have been characterised as linear combinations of at most two k-spaces (Result 3.3). In the general case, only the minimum weight codewords of $\mathcal{C}_{j,k}(n,q)$ have been characterised as scalar multiples the k-spaces (Result 3.1).

Less is known about the dual code $\mathcal{C}_{j,k}(n,q)^\perp$. In general, the minimum weight of $\mathcal{C}_{j,k}(n,q)^\perp$ is not known. However, this minimum weight is at most $2q^{n-k}$; if q is prime, the minimum weight of $\mathcal{C}_{j,j+1}(n,q)^\perp$ is equal to this value and its minimum weight codewords are characterised as being scalar multiples of so-called *standard words* (Definition 3.5, Result 3.6). If q is even, the minimum weight of $\mathcal{C}_{0,k}(n,q)^\perp$ equals $(q+2)q^{n-k-1}$ (Result 3.7).

A further overview of results on these codes can be found in [LSVdV10] and [ADSW20].

2 Outline and main results

As mentioned before, all preliminaries needed to guide you through this article can be found in Section 3.

In Section 4, we study the relation between $\mathcal{C}_{j,k}(n,q)$, $\mathcal{C}_{j,n-k+j}(n,q)^\perp$, their intersection (i.e. the *hull* $\mathcal{H}_{j,k}(n,q)$ of $\mathcal{C}_{j,k}(n,q)$) and their span. We bundle several properties that were already known for specific values of j, k, n and q, and present them in a general context.

In Section 5 and Section 6, we investigate the small weight codewords of $\mathcal{C}_{0,k}(n,q)$ and $\mathcal{C}_{j,k}(n,q)$, respectively. In Section 5, we use the known results concerning small weight codewords of $\mathcal{C}_{0,k}(k+1,q)$ to characterise all codewords of $\mathcal{C}_{0,k}(n,q)$ up to weight $W(k,q)$. The exact value of the latter bound (as well as the meaning of the sets Q_i) can be found in Definition 3.2, but for the sake of simplicity, one can view this bound to be roughly equal to $(3-3/q)q^k$ if q is large enough.

Theorem 5.9. *If c is a codeword of $\mathcal{C}_k(n,q)$, with $\mathrm{wt}(c) \leqslant W(k,q)$, then c is a linear combination of at most two k-spaces. Moreover, if $q \in Q_3 \cup Q_4 \cup Q_5$, then this bound is tight.*

In particular, the minimum weight codewords of the hull $\mathcal{H}_{0,k}(n,q)$ are characterised as well.

Corollary 5.10. *If c is a codeword of $\mathcal{H}_{0,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(k,q)$, then c is a scalar multiple of the difference of two k-spaces. In particular, the minimum weight of $\mathcal{H}_{0,k}(n,q)$ is $2q^k$, and the minimum weight codewords are scalar multiples of the difference of two k-spaces through a common $(k-1)$-subspace.*

These results, in turn, are used in Section 6 as base cases to characterise all codewords of $\mathcal{C}_{j,k}(n,q)$ and $\mathcal{H}_{j,k}(n,q)$ up to weight $W(j,k,q)$. Again, the exact value of the latter bound can be found in Definition 3.4, but it is at least $(3-7/q)\begin{bmatrix}k+1\\j+1\end{bmatrix}_q$ if q is large enough.

Theorem 6.7. *(1) If c is a codeword of $\mathcal{C}_{j,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(j,k,q)$, then c is a linear combination of at most two k-spaces.*

(2) If c is a codeword of $\mathcal{H}_{j,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(j,k,q)$, then c is a scalar multiple of the difference of two k-spaces. In particular, if $q \notin Q_1$, then the minimum weight of $\mathcal{H}_{j,k}(n,q)$ is $2q^{k-j}\begin{bmatrix}k\\j\end{bmatrix}_q$, and the minimum weight codewords are scalar multiples of the difference of two k-spaces through a common $(k-1)$-space.

The following, somewhat weaker result is valid for any prime power q.

Theorem 6.8. *If c is a codeword of $\mathcal{C}_{j,k}(n,q)$, with*

$$\mathrm{wt}(c) \leqslant \frac{2q^k}{\theta_j}\begin{bmatrix}k\\j\end{bmatrix}_q,$$

then c is a scalar multiple of a k-space. As a consequence, the minimum weight of $\mathcal{H}_{j,k}(n,q)$ is larger than $2q^k\begin{bmatrix}k\\j\end{bmatrix}_q/\theta_j$.

As a final note to this section, we investigate the cyclicity of $\mathcal{C}_{j,k}(n,q)$.

Theorem 6.10. *The code $\mathcal{C}_{j,k}(n,q)$ is equivalent to a cyclic code if and only if $j=0$.*

In Section 7, we shift our focus to the dual code $\mathcal{C}_{j,k}(n,q)^\perp$ and manage to reduce both problems of determining its minimum weight and characterising its minimum weight codewords to the codes $\mathcal{C}_{0,1}(n,q)^\perp$. This is done using the construction of a *pull-back* (Construction 7.1). Pull-backs are codewords of $\mathcal{C}_{j,k}(n,q)^\perp$ constructed from codewords of $\mathcal{C}_{0,k-j}(n-j,q)^\perp$.

Theorem 7.8. *If $j>0$, then all minimum weight codewords of $\mathcal{C}_{j,k}(n,q)^\perp$ are pull-backs.*

As a consequence, known results concerning the minimum weight problem of $\mathcal{C}_{j,k}(n,q)^\perp$ (e.g. Result 3.6 and 3.7) are found to be valid for general j and k.

Corollary 7.10. *(1)* $d\left(\mathcal{C}_{j,k}(n,q)^\perp\right) = d\left(\mathcal{C}_{0,1}(n-k+1,q)^\perp\right)$.

(2) If p is prime, then the minimum weight codewords of $\mathcal{C}_{j,k}(n,p)^\perp$ are scalar multiples of the standard words, and thus have weight $2p^{n-k}$.

(3) If q is even, then $d\left(\mathcal{C}_{j,k}(n,q)^\perp\right) = (q+2)q^{n-k-1}$.

In Section 8 we summarise in short what is known about the dimension of these codes. We conclude this article with Section 9 by briefly discussing some open problems concerning this topic.

3 Preliminaries

3.1 Basic notation

Throughout this entire article, we will assume p to be a prime number and $q := p^h$, with $h \in \mathbb{N}^*$. Moreover, we consider natural numbers j, k and n, with the general assumption that

$$0 \leqslant j < k < n.$$

Hence, keep in mind that $k \geqslant 1$ and $n \geqslant 2$.
We will denote the Galois field $\mathrm{GF}(q)$ of order q by $\mathbb{F}_q$ and the Desarguesian projective space of (projective) dimension n over $\mathbb{F}_q$ by $\mathrm{PG}(n,q)$. For any number $m \in \mathbb{N}$, the number of j-spaces in $\mathrm{PG}(m,q)$ is given by the Gaussian coefficient

$$\begin{bmatrix} m+1 \\ j+1 \end{bmatrix}_q := \frac{(q^{m+1}-1)(q^m-1)\cdots(q^{m-j+1}-1)}{(q^{j+1}-1)(q^j-1)\cdots(q-1)}.$$

By convention, we define $\begin{bmatrix} m+1 \\ 0 \end{bmatrix}_q$ to be 1 and we denote $\theta_m := \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q$, with the extension that $\theta_m := 0$ for values $m \in \mathbb{Z} \setminus \mathbb{N}$.
Denote the set of all j-subspaces of a projective space π by $G_j(\pi)$. We denote the latter by $G_j(n,q)$ if π is the ambient space $\mathrm{PG}(n,q)$. If π or n and q are clear from context, we will denote this simply by G_j. Let $V(j,\pi)$ denote the p-ary vector space of functions from $G_j(\pi)$ to $\mathbb{F}_p$, i.e. $V(j,\pi) := \mathbb{F}_p^{G_j(\pi)}$. Similarly, $V(j,n,q) := \mathbb{F}_p^{G_j(n,q)}$. We will denote the functions that map everything to one, respectively zero, by $\mathbf{1}$, respectively $\mathbf{0}$. Moreover, for any $v \in V(j,n,q)$ and any $\lambda \in G_j(n,q)$, the value $v(\lambda)$ will often be described as *the value of λ w.r.t. v*.
We can identify a k-space κ of $\mathrm{PG}(n,q)$ with the function $\kappa^{(j)} \in V(j,n,q)$ such that

$$\kappa^{(j)}(\lambda) = \begin{cases} 1 & \text{if } \lambda \subseteq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

If j is clear from context, we will denote $\kappa^{(j)}$ as κ. There should be no confusion. Let $\mathcal{C}_{j,k}(n,q)$ denote the subspace of $V(j,n,q)$ generated by $G_k(n,q)^{(j)} := \left\{\kappa^{(j)} : \kappa \in G_k(n,q)\right\}$. We will also denote $\mathcal{C}_{0,k}(n,q)$ as $\mathcal{C}_k(n,q)$.
Alternatively, one could define the code $\mathcal{C}_{j,k}(n,q)$ as follows. Consider the p-ary incidence matrix A of k-spaces and j-spaces, i.e. the rows of the matrix correspond to the k-spaces of $\mathrm{PG}(n,q)$ and the columns to the j-spaces. Put a one in the matrix if the j-space corresponding to the column is contained in the k-space corresponding to the row, and zero otherwise. Symbolically,

$$A \in \mathbb{F}_p^{G_k \times G_j} \qquad \text{and} \qquad A_{\kappa,\lambda} = \begin{cases} 1 & \text{if } \lambda \subseteq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

In this way, $\mathcal{C}_{j,k}(n,q)$ is the row span of the matrix A. However, we prefer the definition of $\mathcal{C}_{j,k}(n,q)$ as a vector subspace of $V(j,n,q)$, as this is more convenient for notation.
If $v \in V(j,n,q)$, define the *support* of v as $\operatorname{supp}(v) := \{\lambda \in G_j : v(\lambda) \neq 0\}$ and the *weight* of v as $\operatorname{wt}(v) := |\operatorname{supp}(v)|$. For a vector subspace W of $V(j,n,q)$, let $d(W)$ denote the minimum weight of W, i.e. $d(W) := \min\{\operatorname{wt}(c) : c \in W \setminus \{\mathbf{0}\}\}$. For $0 \leqslant i < j$, we will also make use of the set $\operatorname{supp}_i(c) := \{\iota \in G_i : (\exists \lambda \in \operatorname{supp}(c))(\iota \subset \lambda)\} = \bigcup_{\lambda \in \operatorname{supp}(c)} G_i(\lambda)$.
Define the *scalar product* of two functions $v, w \in V(j,n,q)$ as

$$v \cdot w := \sum_{\lambda \in G_j} v(\lambda) w(\lambda).$$

Define the *dual code* of $\mathcal{C}_{j,k}(n,q)$ as its orthogonal complement with respect to the above scalar product. This means that the dual code is

$$\mathcal{C}_{j,k}(n,q)^{\perp} := \{v \in V(j,n,q) : (\forall c \in \mathcal{C}_{j,k}(n,q))(c \cdot v = 0)\}.$$

Define the *hull* $\mathcal{H}_{j,k}(n,q)$ of $\mathcal{C}_{j,k}(n,q)$ as

$$\mathcal{H}_{j,k}(n,q) := \mathcal{C}_{j,k}(n,q) \cap \mathcal{C}_{j,n-k+j}(n,q)^{\perp}.$$

3.2 Known results and the bounds $W(k,q)$ and $W(j,k,q)$

Some important characterisations are already known.

Result 3.1 ([BI02, Theorem 1]). *The minimum weight of $\mathcal{C}_{j,k}(n,q)$ is $\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q$, and minimum weight codewords are scalar multiples of k-spaces, i.e. scalar multiples of the elements of $G_k(n,q)^{(j)}$.*

If $j = 0$, stronger characterisations are known.

Definition 3.2. Define $W(k,q)$ as

$$W(k,q) := \begin{cases} 2q^k & \text{if } q \in Q_1 := \{q \,:\, q \leqslant 9\} \cup \{16, 25, 27, 49\}, \\ 2\theta_k & \text{if } q \in Q_2 := \{q \,:\, 9 < q \leqslant 23, q \neq 16\} \cup \{29, 31, 32, 121\}, \\ 3q^k - 3q^{k-1} - 1 & \text{if } q \in Q_3 := \{q \,:\, q > 32,\ q \text{ prime}\}, \\ 3q^k - 3q^{k-1} + \theta_{k-2} - 1 & \text{if } q \in Q_4 := \{q \,:\, q > 32,\ q \text{ even}\}, \\ 3q^k - 2q^{k-1} + \theta_{k-2} - 1 & \text{if } q \in Q_5, \text{ the complement of } \bigcup_{i=1}^4 Q_i. \end{cases}$$

We will use the following weakened version of known characterisations.

Result 3.3 ([ADSW20, Corollary 2.2.13] [PZ18, Theorem 1.4]). *If c is a codeword of $\mathcal{C}_k(k+1,q)$, with $\operatorname{wt}(c) \leqslant W(k,q)$, then c is a linear combination of at most two k-spaces. Moreover, this bound is tight if $q \in Q_3 \cup Q_4 \cup Q_5$.*

In Section 5 we prove that this holds for all codes $\mathcal{C}_k(n,q)$.

Definition 3.4. Define $W(j,k,q)$ as

$$W(j,k,q) := \begin{cases} \frac{2q^k}{\theta_j} \begin{bmatrix} k \\ j \end{bmatrix}_q & \text{if } q \in Q_1, \\ 2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q & \text{if } q \in Q_2, \\ \left(3 - \frac{7}{q}\right) \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q & \text{if } q \in Q_3 \cup Q_4, \\ \left(3 - \frac{6}{q}\right) \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q & \text{if } q \in Q_5. \end{cases}$$

Remark that $W(0,k,q) \leqslant W(k,q)$. The focus of Section 6 are Theorems 6.7 and 6.8, where we prove that codewords of $\mathcal{C}_{j,k}(n,q)$ up to weight $W(j,k,q)$ are linear combinations of at most two k-spaces.

Definition 3.5. Let ι be a $(j-1)$-space, and let π and ρ be two $(n-k+j)$-spaces through an $(n-k+j-1)$-space containing ι. Define $v \in V(j,n,q)$ as

$$v := \sum_{\substack{\lambda \in G_j(\pi) \\ \iota \subset \lambda}} \lambda^{(j)} - \sum_{\substack{\lambda' \in G_j(\rho) \\ \iota \subset \lambda'}} \lambda'^{(j)}.$$

Codewords of this form are called *standard words* of $\mathcal{C}_{j,k}(n,q)^\perp$.

Result 3.6 ([BI02, Theorem 3, Proposition 2])**.** *Standard words of $\mathcal{C}_{j,k}(n,q)^\perp$ are codewords of $\mathcal{C}_{j,k}(n,q)^\perp$ of weight $2q^{n-k}$. Therefore, the minimum weight of $\mathcal{C}_{j,k}(n,q)^\perp$ is at most $2q^{n-k}$. Moreover, if p is prime, then the minimum weight codewords of $\mathcal{C}_{j,j+1}(n,p)^\perp$ are the scalar multiples of the standard words.*

Result 3.7 ([CKdR99, Theorem 1])**.** *If q is even, then $d\left(\mathcal{C}_k(n,q)^\perp\right) = (q+2)q^{n-k-1}$.*

4 A brief note on the relation with the dual code

As a generalisation of [AK92, Chapter 6] and [LSVdV08, Lemma 2], we have the following.

Lemma 4.1. *(1) If $c \in \mathcal{C}_{j,k}(n,q)$, then $c \cdot \pi$ is equal for all subspaces π in $\mathrm{PG}(n,q)$ with $\dim(\pi) \geqslant n-k+j$.*

(2) $\mathcal{H}_{j,k}(n,q) = \{c \in \mathcal{C}_{j,k}(n,q) : c \cdot \mathbf{1} = 0\} = \langle \kappa - \kappa' : \kappa \in G_k \rangle$ for any $\kappa' \in G_k$.

(3) $\dim\left(\mathcal{H}_{j,k}(n,q)\right) = \dim\left(\mathcal{C}_{j,k}(n,q)\right) - 1$.

Proof. (1) Take a k-space κ and a subspace π with $\dim(\pi) \geqslant n-k+j$. It is easy to see that $\kappa^{(j)} \cdot \pi^{(j)}$ equals the number of j-spaces in $\kappa \cap \pi$ modulo p. By Grassmann's identity, $\dim(\kappa \cap \pi) \geqslant \dim(\kappa) + \dim(\pi) - n \geqslant j$. Therefore, the number of j-spaces in $\kappa \cap \pi$ equals $\begin{bmatrix} \dim(\kappa\cap\pi)+1 \\ j+1 \end{bmatrix}_q \equiv 1 \pmod p$. Now take a codeword $c \in \mathcal{C}_{j,k}(n,q)$. Then c is a linear combination of k-spaces, so $c = \sum_i \alpha_i \kappa_i$ for some $\alpha_i \in \mathbb{F}_p$ and $\kappa_i \in G_k$. Since the scalar product is bilinear, we have that

$$c \cdot \pi = \left(\sum_i \alpha_i \kappa_i\right) \cdot \pi = \sum_i \alpha_i (\kappa_i \cdot \pi) = \sum_i \alpha_i,$$

hence $c \cdot \pi$ is equal for all π.

(2, 3) Take a codeword $c \in \mathcal{C}_{j,k}(n,q)$. Then $c \in \mathcal{C}_{j,n-k+j}(n,q)^\perp$ if and only if c is orthogonal to all codewords of $\mathcal{C}_{j,n-k+j}(n,q)$. Since the scalar product is bilinear, is suffices that c is orthogonal to the generators of $\mathcal{C}_{j,n-k+j}(n,q)$. By (1), this only requires that the scalar product of c with a specific subspace of dimension at least $n-k+j$ is zero, e.g. the whole space. This means that $c \cdot \mathbf{1}$ is zero. Hence, $\mathcal{H}_{j,k}(n,q) = \{c \in \mathcal{C}_{j,k}(n,q) : c \cdot \mathbf{1} = 0\}$.
Since $c \cdot \mathbf{1} = 0$ is a linear equation, we know that $\{c \in \mathcal{C}_{j,k}(n,q) : c \cdot \mathbf{1} = 0\}$ is a vector subspace of $\mathcal{C}_{j,k}(n,q)$ of codimension 0 or 1. Since we have proven in (1) that, for any k-space κ, $\kappa \cdot \mathbf{1} = 1$, this vector subspace must be a proper subspace, hence it has codimension 1, proving (3).
Now take two k-spaces κ and κ'. It is clear that $\kappa - \kappa' \in \mathcal{C}_{j,k}(n,q)$. If $\pi \in G_{n-k+j}$, then we know that $\kappa \cdot \pi = \kappa' \cdot \pi = 1$ by (1). Hence, $\pi \cdot (\kappa - \kappa') = 0$. Therefore, $\kappa - \kappa'$ is orthogonal to all generators of $\mathcal{C}_{j,n-k+j}(n,q)$, which means that $\kappa - \kappa' \in \mathcal{C}_{j,n-k+j}(n,q)^\perp$. As a result, if we fix $\kappa' \in G_k$, $K := \langle \kappa - \kappa' : \kappa \in G_k \rangle \leqslant \mathcal{H}_{j,k}(n,q)$. Since $K \oplus \langle \kappa' \rangle = \mathcal{C}_{j,k}(n,q)$, the codimension of K in $\mathcal{C}_{j,k}(n,q)$ is at most one. Thus, $\dim(K) \geqslant \dim\left(\mathcal{H}_{j,k}(n,q)\right)$. This is only possible if those spaces coincide. ☐

We can also say something about the code $\mathcal{S}_{j,k}(n,q) := \left\langle \mathcal{C}_{j,k}(n,q), \mathcal{C}_{j,n-k+j}(n,q)^\perp \right\rangle$.

Lemma 4.2. *(1) $\dim\left(\mathcal{S}_{j,k}(n,q)\right) = \dim\left(\mathcal{C}_{j,n-k+j}(n,q)^\perp\right) + 1$.*

(2) $\mathcal{S}_{j,k}(n,q) = \mathcal{H}_{j,n-k+j}(n,q)^\perp = \{v \in V(j,n,q) : (\exists \alpha \in \mathbb{F}_p)(\forall \kappa \in G_{n-k+j})(v \cdot \kappa = \alpha)\}$.

(3) The minimum weight codewords of $\mathcal{S}_{0,k}(n,q)$ *are scalar multiples of* k*-spaces.*

(4) If $j \geqslant 1$*, then the minimum weight codewords of* $\mathcal{S}_{j,k}(n,q)$ *lie in* $\mathcal{C}_{j,n-k+j}(n,q)^\perp$*.*

Proof. (1) By Grassmann's identity and Lemma 4.1 (3), we have

$$\begin{aligned}\dim\left(\mathcal{S}_{j,k}(n,q)\right) &= \dim\left(\mathcal{C}_{j,k}(n,q)\right) + \dim\left(\mathcal{C}_{j,n-k+j}(n,q)^\perp\right) - \dim\left(\mathcal{C}_{j,k}(n,q) \cap \mathcal{C}_{j,n-k+j}(n,q)^\perp\right)\\ &= \dim\left(\mathcal{C}_{j,n-k+j}(n,q)^\perp\right) + 1.\end{aligned}$$

(2) Since $\langle A, B\rangle^\perp = A^\perp \cap B^\perp$, we have that $\mathcal{S}_{j,k}(n,q)^\perp = \mathcal{C}_{j,k}(n,q)^\perp \cap \mathcal{C}_{j,n-k+j}(n,q) = \mathcal{H}_{j,n-k+j}(n,q)$. By Lemma 4.1 (2), this means that $\mathcal{S}_{j,k}(n,q)^\perp = \langle \kappa - \kappa' : \kappa, \kappa' \in G_{n-k+j}\rangle^\perp$. Hence, $v \in \mathcal{S}_{j,k}(n,q) \Leftrightarrow (\forall \kappa, \kappa' \in G_{n-k+j})(v \cdot (\kappa - \kappa') = 0)$. This means that $v \in \mathcal{S}_{j,k}(n,q)$ if and only if $v \cdot \kappa$ is equal for all $(n-k+j)$-spaces κ.

(3) The arguments used in the literature to prove this exact same statement about $\mathcal{C}_k(n,q)$ are also valid for the bigger code $\mathcal{S}_{0,k}(n,q)$; for instance, see [BI02, Proposition 1], where the authors make the exact same observation at the very end of their work.

(4) Assume that $j \geqslant 1$ and take a codeword $c \in \mathcal{S}_{j,k}(n,q)$, with $c \notin \mathcal{C}_{j,n-k+j}(n,q)^\perp$. Then we know that there exists some $\alpha \in \mathbb{F}_p^*$, with $c \cdot \kappa = \alpha$, for all $\kappa \in G_{n-k+j}$. In particular, this means that every $(n-k+j)$-space κ contains an element of $\mathrm{supp}(c)$. Consider the set $V = \{(\lambda, \kappa) : \lambda \in \mathrm{supp}(c), \lambda \subset \kappa \in G_{n-k+j}\}$. Since for every κ, there exists a λ with $(\lambda,\kappa) \in V$, we get

$$\mathrm{wt}(c)\begin{bmatrix} n-j \\ k-j\end{bmatrix}_q = \mathrm{wt}(c)\begin{bmatrix} n-j \\ (n-k+j)-j\end{bmatrix}_q = |V| \geqslant \begin{bmatrix} n+1 \\ (n-k+j)+1\end{bmatrix}_q = \begin{bmatrix} n+1 \\ k-j\end{bmatrix}_q.$$

Here we used the fact that $\begin{bmatrix} n \\ k\end{bmatrix}_q = \begin{bmatrix} n \\ n-k\end{bmatrix}_q$. Manipulating this inequality yields

$$\begin{aligned}\mathrm{wt}(c) \geqslant \frac{\begin{bmatrix} n+1 \\ k-j\end{bmatrix}_q}{\begin{bmatrix} n-j \\ k-j\end{bmatrix}_q} &= \frac{\frac{(q^{n+1}-1)(q^n-1)\cdots(q^{n+2-k+j}-1)}{(q^{k-j}-1)(q^{k-j-1}-1)\cdots(q-1)}}{\frac{(q^{n-j}-1)(q^{n-j-1}-1)\cdots(q^{n-k+1}-1)}{(q^{k-j}-1)(q^{k-j-1}-1)\cdots(q-1)}} = \frac{q^{n+1}-1}{q^{n-j}-1}\frac{q^n-1}{q^{n-j-1}-1}\cdots\frac{q^{n+2-k+j}-1}{q^{n-k+1}-1}\\ &> (q^{j+1})^{k-j} \geqslant 2q^{k-j}.\end{aligned}$$

However, by Result 3.6, the minimum weight of $\mathcal{C}_{j,n-k+j}(n,q)^\perp$ is at most $2q^{k-j}$. Hence, the minimum weight codewords of $\mathcal{S}_{j,k}(n,q)$ must be contained in $\mathcal{C}_{j,n-k+j}(n,q)^\perp$. □

Also note that, given a space π with $\dim(\pi) > k$, $\pi^{(j)} = \sum_{\kappa \in G_k(\pi)} \kappa^{(j)}$. This way, we see that if $k > k'$, then $\mathcal{C}_{j,k}(n,q) \leqslant \mathcal{C}_{j,k'}(n,q)$ and $\mathcal{C}_{j,k}(n,q)^\perp \geqslant \mathcal{C}_{j,k'}(n,q)^\perp$.

5 Codes of points and k-spaces

The tool to guide us towards a characterisation of small weight codewords of $\mathcal{C}_k(n,q)$, is the following linear map. It is essentially due to Lavrauw, Storme & Van de Voorde [LSVdV08, Lemma 11], but they only use it for a result regarding $\mathcal{C}_k(n,q)^\perp$ (see Result 7.9). We define it in a more general form, for all values of j.

Definition 5.1. Take a point R in $\mathrm{PG}(n,q)$ and a hyperplane π not through R. Take an integer $j \leqslant n-2$ and a function $v \in V(j,n,q)$. Then we define the function $\mathrm{proj}^{(j)}_{R,\pi}(v)$ in $V(j,\pi)$ by

$$\mathrm{proj}^{(j)}_{R,\pi}(v) : \lambda \mapsto \sum_{\lambda' \in G_j(\langle R, \lambda\rangle)} v(\lambda').$$

This means that the value of a j-space $\lambda \subset \pi$ w.r.t. $\mathrm{proj}_{R,\pi}^{(j)}(v)$ is the sum of the values w.r.t. c of all j-spaces λ' in the $(j+1)$-space $\langle R, \lambda\rangle$. We could also write this as

$$\mathrm{proj}_{R,\pi}^{(j)}(v)(\lambda) = v \cdot \langle R, \lambda\rangle^{(j)} .$$

We view $\mathrm{proj}_{R,\pi}^{(j)} : v \mapsto \mathrm{proj}_{R,\pi}^{(j)}(v)$ as a mapping from $V(j,n,q)$ to $V(j,\pi)$. If $j = 0$, we will denote $\mathrm{proj}_{R,\pi}^{(0)}$ by $\mathrm{proj}_{R,\pi}$.

We now present the most important properties of this map.

Lemma 5.2. *Assume that R is a point of $\mathrm{PG}(n,q)$ and that π is a hyperplane not through R. Then the following holds:*

(1) The map $\mathrm{proj}_{R,\pi}^{(j)}$ is linear.

(2) If $k < n-1$, then $\mathrm{proj}_{R,\pi}^{(j)}(\mathcal{C}_{j,k}(n,q)) = \mathcal{C}_{j,k}(n-1,q)$.

(3) If $k > j+1$, then $\mathrm{proj}_{R,\pi}^{(j)}(\mathcal{C}_{j,k}(n,q)^{\perp}) = \mathcal{C}_{j,k-1}(n-1,q)^{\perp}$.

(4) If $v \in V(j,n,q)$ and $R \notin \mathrm{supp}_0(v)$, then $\mathrm{wt}(\mathrm{proj}_{R,\pi}^{(j)}(v)) \leqslant \mathrm{wt}(v)$, with equality if and only if no $(j+1)$-space through R contains more than one j-space of $\mathrm{supp}(v)$.

(5) If $v \in V(j,n,q)$, then $v \cdot \mathbf{1} = \mathrm{proj}_{R,\pi}^{(j)}(v) \cdot \mathbf{1}$.

Proof. (1) To prove that $\mathrm{proj}_{R,\pi}^{(j)}$ is linear, we take $\alpha, \beta \in \mathbb{F}_p$, and $v, w \in V(j,n,q)$. We need to prove that $\mathrm{proj}_{R,\pi}^{(j)}(\alpha v + \beta w) = \alpha \mathrm{proj}_{R,\pi}^{(j)}(v) + \beta \mathrm{proj}_{R,\pi}^{(j)}(w)$. Take a j-space $\lambda \subset \pi$. Then

$$\begin{aligned}\mathrm{proj}_{R,\pi}^{(j)}(\alpha v + \beta w)(\lambda) &= (\alpha v + \beta w)\cdot \langle R,\lambda\rangle = \alpha v \cdot \langle R,\lambda\rangle + \beta w \cdot \langle R,\lambda\rangle \\ &= \alpha \mathrm{proj}_{R,\pi}^{(j)}(v)(\lambda) + \beta \mathrm{proj}_{R,\pi}^{(j)}(w)(\lambda).\end{aligned}$$

Since this holds for every j-space $\lambda \subset \pi$, this means that $\mathrm{proj}_{R,\pi}^{(j)}(\alpha v + \beta w) = \alpha \mathrm{proj}_{R,\pi}^{(j)}(v) + \beta \mathrm{proj}_{R,\pi}^{(j)}(w)$.

(2) Let κ be a k-space of $\mathrm{PG}(n,q)$. First, assume that $R \notin \kappa$. It is easy to see that $\mathrm{proj}_{R,\pi}^{(j)}(\kappa)$ is the k-space $\langle R,\kappa\rangle \cap \pi$. So assume that $R \in \kappa$. Take a j-space $\lambda \subset \pi$. Then $\mathrm{proj}_{R,\pi}^{(j)}(\kappa)(\lambda)$ equals the number of j-spaces in $\langle R,\lambda\rangle \cap \kappa$. Note that $\dim\big(\langle R,\lambda\rangle\cap\kappa\big) = \dim(\lambda\cap\kappa)+1$. This implies that

$$\mathrm{proj}_{R,\pi}^{(j)}(\kappa)(\lambda) = \begin{cases} 1 & \text{if } \dim(\lambda\cap\kappa) \geqslant j-1, \\ 0 & \text{otherwise.}\end{cases}$$

The number of k-spaces κ' in π through a j-space λ, containing the $(k-1)$-space $\kappa\cap\pi$ equals 0 if $\dim(\lambda\cap\kappa) < j-1$, equals 1 if $\dim(\lambda\cap\kappa) = j-1$, and equals $\left[{(n-1)-(k-1) \atop k-(k-1)}\right]_q \equiv 1 \pmod p$ if $\dim(\lambda\cap\kappa) = j$. Thus,

$$\mathrm{proj}_{R,\pi}^{(j)}(\kappa) = \sum_{\substack{\kappa' \in G_k(\pi) \\ \kappa\cap\pi\subset\kappa'}} \kappa' \in \mathcal{C}_{j,k}(n-1,q).$$

Therefore the map $\mathrm{proj}_{R,\pi}^{(j)}$ maps the set $G_k(n,q)^{(j)}$, which generates the code $\mathcal{C}_{j,k}(n,q)$, to a subset of $\mathcal{C}_{j,k}(n-1,q)$, containing its generating set $G_k(\pi)^{(j)}$. Since this map is linear, this proves that $\mathrm{proj}_{R,\pi}^{(j)}(\mathcal{C}_{j,k}(n,q)) = \mathcal{C}_{j,k}(n-1,q)$.

(3) Take $c \in \mathcal{C}_{j,k}(n,q)^\perp$. To prove that $\mathrm{proj}^{(j)}_{R,\pi}(c) \in \mathcal{C}_{j,k-1}(n-1,q)^\perp$, we need to prove that $\mathrm{proj}^{(j)}_{R,\pi}(c) \cdot \kappa = 0$ for every $(k-1)$-space $\kappa \subset \pi$.

$$\begin{aligned}\mathrm{proj}^{(j)}_{R,\pi}(c) \cdot \kappa = \sum_{\lambda \in G_j(\pi)} \mathrm{proj}^{(j)}_{R,\pi}(c)(\lambda) \cdot \kappa(\lambda) &= \sum_{\substack{\lambda \in G_j(\pi) \\ \lambda \subset \kappa}} \sum_{\lambda' \in G_j(\langle R,\lambda\rangle)} c(\lambda') \\ &= \sum_{\lambda' \in G_j(\langle R,\kappa\rangle)} c(\lambda') \sum_{\substack{\lambda \in G_j(\kappa) \\ \lambda' \subset \langle R,\lambda\rangle}} 1.\end{aligned}$$

For a fixed j-space λ' in $\langle R,\kappa\rangle$, we have

$$\sum_{\substack{\lambda \in G_j(\kappa) \\ \lambda' \subset \langle R,\lambda\rangle}} 1 = \begin{cases} 1 & \text{if } R \notin \lambda', \\ \theta_{k-j-1} & \text{otherwise} \end{cases} \equiv 1 \pmod p.$$

Therefore,

$$\mathrm{proj}^{(j)}_{R,\pi}(c) \cdot \kappa = \sum_{\lambda' \in G_j(\langle R,\kappa\rangle)} c(\lambda') = c \cdot \langle R,\kappa\rangle = 0,$$

because $\langle R,\kappa\rangle$ is a k-space and $c \in \mathcal{C}_{j,k}(n,q)^\perp$. Hence, $\mathrm{proj}^{(j)}_{R,\pi}(\mathcal{C}_{j,k}(n,q)^\perp) \leqslant \mathcal{C}_{j,k-1}(n-1,q)^\perp$. To prove that equality holds, we can embed a codeword c' of $\mathcal{C}_{j,k-1}(n-1,q)^\perp$ in π (see Construction 7.6). The image of this embedded codeword under $\mathrm{proj}^{(j)}_{R,\pi}$ will again be c'.

(4) It holds that if $\lambda \in \mathrm{supp}(\mathrm{proj}^{(j)}_{R,\pi}(v))$, then the $(j+1)$-space $\langle R,\lambda\rangle$ must contain a j-space of $\mathrm{supp}(v)$. Hence, if $R \notin \mathrm{supp}_0(v)$, every j-space in $\mathrm{supp}(c)$ lies in a unique $(j+1)$-space through R, which implies that the number of $(j+1)$-spaces through R that contain an element of $\mathrm{supp}(v)$ is at most $\mathrm{wt}(v)$. Thus, $\mathrm{wt}(\mathrm{proj}^{(j)}_{R,\pi}(v)) \leqslant \mathrm{wt}(v)$. It is easy to see that equality holds if and only if no $(j+1)$-space through R contains more than one element of $\mathrm{supp}(v)$.

(5)

$$\begin{aligned}\mathrm{proj}^{(j)}_{R,\pi}(v) \cdot \mathbf{1} &= \sum_{\lambda \in G_j(\pi)} \mathrm{proj}_{R,\pi}(v)(\lambda) \cdot 1 = \sum_{\lambda \in G_j(\pi)} \sum_{\lambda' \in G_j(\langle R,\lambda\rangle)} v(\lambda') = \sum_{\lambda' \in G_j(n,q)} v(\lambda') \sum_{\substack{\lambda \in G_j(\pi) \\ \lambda' \subset \langle R,\lambda\rangle}} 1 \\ &= \sum_{\substack{\lambda' \in G_j(n,q) \\ R \notin \lambda'}} v(\lambda') + \begin{bmatrix} (n-1)-(j-1) \\ j-(j-1) \end{bmatrix}_q \sum_{\substack{\lambda' \in G_j(n,q) \\ R \in \lambda'}} v(\lambda') \\ &\equiv \sum_{\substack{\lambda' \in G_j(n,q) \\ R \notin \lambda'}} v(\lambda') + \sum_{\substack{\lambda' \in G_j(n,q) \\ R \in \lambda'}} v(\lambda') = v \cdot \mathbf{1} \pmod p.\end{aligned}$$

□

Remark 5.3. When constructing $\mathrm{proj}_{R,\pi}(c)$, what we are actually doing is projecting from the point R onto a hyperplane π. One could also view this as working in the quotient geometry of $\mathrm{PG}(n,q)$ through R. This way we see that the choice of π is not really relevant. In other words, for any two choices of hyperplanes $\pi_1, \pi_2 \not\ni R$ in $\mathrm{PG}(n,q)$, the nature of the codewords $\mathrm{proj}_{R,\pi_1}(c)$ and $\mathrm{proj}_{R,\pi_2}(c)$ will essentially stay the same. More rigorously, there exists a collineation β from π_1 to π_2 such that $\mathrm{proj}_{R,\pi_1}(c)(\lambda) = \mathrm{proj}_{R,\pi_2}(c)(\lambda^\beta)$, for every $\lambda \in G_j(\pi_1)$. This collineation β maps a subspace λ of π_1 to $\langle R,\lambda\rangle \cap \pi_2$. The reason that we emphasize which hyperplane is considered is solely to obtain a natural embedding of $\mathrm{supp}(\mathrm{proj}_{R,\pi}(c))$ in $\mathrm{PG}(n-1,q)$.
Therefore, when considering $\mathrm{proj}_{R,\pi}(c)$, we can, at any time and w.l.o.g., choose π to be any other hyperplane not containing R.

Eventually, we will use this map to characterise small weight codewords of $\mathcal{C}_k(n,q)$. However, we first need a few important lemmas, some of which are tedious to prove.

Lemma 5.4. *Let $c \in \mathcal{C}_k(n,q)$ be a linear combination of three k-spaces, which can't be written as a linear combination of at most two k-spaces. Then $\mathrm{wt}(c) > W(k,q)$.*

Proof. Let us denote these three distinct k-spaces by κ_i $(i = 1,2,3)$. We write $\sigma := \bigcap_{i=1}^{3} \kappa_i$, $K := \langle \kappa_1, \kappa_2, \kappa_3 \rangle$, and $s := \dim(\sigma)$. A simple but tedious argument to prove this result is finding a lower bound on $\mathrm{wt}(c)$ that exceeds $W(k,q)$. This is done by counting points that lie in precisely one of the three k-spaces κ_i, as these points are necessarily contained in $\mathrm{supp}(c)$. As the proof involves a case-by-case analysis of the geometric nature of these k-spaces, we will omit most details of the easier cases.
If $s = k-1$, one can prove rather easily that $\mathrm{wt}(c) \in \left\{3q^k, 3q^k + \theta_{k-1}\right\}$.
If $s = k-2$, there are two cases to consider. In the first case, we assume that two k-spaces intersect in σ. Hence, each of these two k-spaces contains at least $\theta_k - \theta_{k-1}$ points not lying in any other of the three spaces. As the third space adds at least $\theta_k - \theta_{k-1} - (\theta_{k-1} - \theta_{k-2})$ points of $\mathrm{supp}(c)$ we haven't considered before, we obtain $\mathrm{wt}(c) \geqslant 3q^k - q^{k-1}$. In the second case, we assume that each two k-spaces intersect in a $(k-1)$-space. As a consequence, either these three k-spaces pairwise intersect in σ, or K is a $(k+1)$-space. As $s < k-1$, we conclude that the latter holds. Hence, we can consider the restriction of the codeword c to K and rely on Result 3.3.
Finally, assume that $s \leqslant k-3$. Denote $\sigma_2 = \kappa_1 \cap \kappa_2$ and $\sigma_3 = \kappa_1 \cap \kappa_3$. We know that $\dim(\sigma_2 \cap \sigma_3) = \dim(\sigma) = s$, and that $\dim(\langle \sigma_2, \sigma_3 \rangle) \leqslant \dim(\kappa_1) = k$. Grassmann's identity implies that $\dim(\sigma_2) + \dim(\sigma_3) \leqslant k + s$. We also know that the dimension of σ_2 and σ_3 are at most $k-1$. Note that if $a \geqslant b$, then $\theta_a + \theta_b < \theta_{a+1} + \theta_{b-1}$. Keeping this in mind, together with $\dim(\sigma_2) + \dim(\sigma_3) \leqslant k + s$, we know that $\sigma_2 \cup \sigma_3$ contains at most $\theta_{k-1} + \theta_{s+1} - \theta_s = \theta_{k-1} + q^{s+1} \leqslant \theta_{k-1} + q^{k-2}$ points. Hence, κ_1 contains at least $\theta_k - \theta_{k-1} - q^{k-2} = q^k - q^{k-2}$ points outside of $\kappa_2 \cup \kappa_3$. Repeating this argument for each of the two other k-spaces, we obtain $\mathrm{wt}(c) \geqslant 3(q^k - q^{k-2})$. □

Definition 5.5. Let S be a point set in $\mathrm{PG}(n,q)$. If a line l of $\mathrm{PG}(n,q)$ intersects S in at most 2 points, we will call l a *short secant* to S. If l intersects S in at least q points, we will call l a *long secant* to S.

The next lemmata make the mild assumption that q is at least 4 or 5. When characterising small weight codewords of $\mathcal{C}_k(n,q)$, the small values of q will be dealt with separately.

Lemma 5.6. *Let c be a codeword of $\mathcal{C}_k(n,q)$ with $q \geqslant 5$ and $\mathrm{wt}(c) \leqslant W(k,q)$.*

(1) All lines in $\mathrm{PG}(n,q)$ are either short or long secants to $\mathrm{supp}(c)$.

(2) $c \cdot s = \begin{cases} c \cdot \mathbf{1} & \text{if } s \text{ is a 2-secant to } \mathrm{supp}(c), \\ 0 & \text{if } s \text{ is a } q\text{-secant to } \mathrm{supp}(c). \end{cases}$

Proof. We will prove this by induction on n. If $n = k+1$, then we know, by Result 3.3, that c is a linear combination of at most two k-spaces. In particular, this implies that $\mathrm{supp}(c)$ is either equal to the empty set, a k space, or the union or symmetric difference of two k-spaces, proving the first statement of the lemma. If s is a 2-secant to $\mathrm{supp}(c)$, then c must be a linear combination of precisely two k-spaces. Then both $c \cdot s$ and $c \cdot \mathbf{1}$ equal the sum of the coefficients arising from this linear combination. If s is a q-secant to $\mathrm{supp}(c)$, then c must be a scalar multiple of the difference of two distinct k-spaces. A q-secant can only exist in this setting if c takes the same non-zero value at all but one point of s. Hence, $c \cdot s = 0$, proving the second statement.
Therefore, let us assume that $n \geqslant k+2$ and that the lemma is true for all codewords in $\mathcal{C}_k(n-1,q)$ with weight at most $W(k,q)$. Note that, by Lemma 5.2 (4), the induction hypothesis implies

that both statements of this lemma hold for the codeword $\mathrm{proj}_{R,\pi}(c)$, for any point $R \notin \mathrm{supp}(c)$ and any hyperplane $\pi \not\ni R$.
Suppose that s is an m-secant to $\mathrm{supp}(c)$ and suppose that every plane through s intersects $\mathrm{supp}(c)$ in at least $m+3$ points. Then $\mathrm{wt}(c) \geqslant 3\theta_{n-2} + m \geqslant 3\theta_k > W(k,q)$, a contradiction. Hence, there exists a plane σ such that $|\sigma \cap \mathrm{supp}(c)| \leqslant m+2$. Let π be a hyperplane intersecting σ in s.

(1) Let $3 \leqslant m \leqslant q-1$. To find a contradiction and prove the first part of the lemma, we distinguish three cases depending on the value of $|\sigma \cap \mathrm{supp}(c)| \in \{m, m+1, m+2\}$. For each of these cases, one can find a point $R \in \sigma \setminus s$ such that s contains precisely m or $m+1$ points (if $m \neq q-1$), or m or $m-1$ points (if $m \neq 3$) of $\mathrm{supp}(\mathrm{proj}_{R,\pi}(c))$. Hence, each of these cases results in the existence of a secant to $\mathrm{supp}(\mathrm{proj}_{R,\pi}(c))$ that is neither short nor long, contradicting the induction hypothesis. We leave the rather tedious details of this case-by-case proof to the reader.

(2) Let $m \in \{2, q\}$. The proof of the second statement can easily be obtained if we know that $\sigma \cap \mathrm{supp}(c) \subseteq s$. Indeed, if this holds, then s is an m-secant to $\mathrm{supp}(\mathrm{proj}_{R,\pi}(c))$ for any choice of $R \in \sigma \setminus s$. Moreover, as all lines through R in σ contain at most one point of $\mathrm{supp}(c)$, we know that $c \cdot s = \mathrm{proj}_{R,\pi}(c) \cdot s$. By the induction hypothesis and Lemma 5.2 (5), we know that

$$\mathrm{proj}_{R,\pi}(c) \cdot s = \begin{cases} \mathrm{proj}_{R,\pi}(c) \cdot \mathbf{1} = c \cdot \mathbf{1} & \text{if } s \text{ is a 2-secant to } \mathrm{supp}(c), \\ 0 & \text{if } s \text{ is a } q\text{-secant to } \mathrm{supp}(c). \end{cases}$$

So let us assume, on the contrary, that $|\sigma \cap \mathrm{supp}(c)| \in \{m+1, m+2\}$.
If $m = 2$, we can find a point $R \in \sigma \setminus (s \cup \mathrm{supp}(c))$ such that s contains precisely $|\sigma \cap \mathrm{supp}(c)| < q$ points of $\mathrm{supp}(\mathrm{proj}_{R,\pi}(c))$, contradicting the assumptions.
Let $m = q$ and let O be the unique point in $s \setminus \mathrm{supp}(c)$. Let t be a line of σ through O containing a point of $(\sigma \cap \mathrm{supp}(c)) \setminus s$. Then all points of $(\sigma \cap \mathrm{supp}(c)) \setminus s$ have to lie on t, as else we can find a 3-secant to $\mathrm{supp}(c)$ in σ, contradicting (1). In this way, if we choose $Q \in t \cap \mathrm{supp}(c)$, QP is a 2-secant to $\mathrm{supp}(c)$ for every choice of $P \in s \setminus \{O\}$. As we already proved the statement of the lemma concerning 2-secants, we know that all values $c \cdot QP$ are the same, for every choice of $P \in s \setminus \{O\}$. As $c \cdot QP = c(Q) + c(P)$, this means that c takes the same value at every point of $s \setminus \{O\}$, resulting in $c \cdot s = 0$. □

Lemma 5.7. *Assume that $\mathcal{S}$ is a point set in $\mathrm{PG}(n,q)$, $q \geqslant 4$, with the property that every line intersects $\mathcal{S}$ in 0, 1, q or $q+1$ points. Then there exists a hyperplane H in $\mathrm{PG}(n,q)$ such that either $\mathcal{S} \subseteq H$ or $\mathcal{S}^c \subseteq H$, where $\mathcal{S}^c$ denotes the complement of $\mathcal{S}$ in $\mathrm{PG}(n,q)$.*

Proof. We prove this by induction on n. Note that it is trivial for $n = 1$. Now assume that it holds in $\mathrm{PG}(n-1,q)$, we will prove that it holds in $\mathrm{PG}(n,q)$. The induction hypothesis implies that for every hyperplane π of $\mathrm{PG}(n,q)$, either $\mathcal{S} \cap \pi$ or $\mathcal{S}^c \cap \pi$ is contained in an $(n-2)$-space of π. If $\mathcal{S}$ spans $\mathrm{PG}(n,q)$, then we can take a hyperplane π spanned by n points of $\mathcal{S}$ and a point $P \in \mathcal{S} \setminus \pi$. By the induction hypothesis, $\mathcal{S}^c \cap \pi$ is contained in an $(n-2)$-space in π. Therefore, there are at least q^{n-1} lines through P intersecting π in a point of $\mathcal{S}$. These lines contain at least q points of $\mathcal{S}$, yielding that $|\mathcal{S}| \geqslant q^{n-1}(q-1)+1$. Note that this lemma is self-complementary in the sense that if we replace $\mathcal{S}$ by $\mathcal{S}^c$, the lemma stays the same. Thus, if $\mathcal{S}^c$ spans $\mathrm{PG}(n,q)$, then $|\mathcal{S}^c| \geqslant q^{n-1}(q-1)+1$. Hence, if both $\mathcal{S}$ and $\mathcal{S}^c$ span $\mathrm{PG}(n,q)$, then

$$\theta_n = |\mathcal{S}| + |\mathcal{S}^c| \geqslant 2(q^{n-1}(q-1)+1),$$

a contradiction if $q \geqslant 4$. Therefore, either $\mathcal{S}$ or $\mathcal{S}^c$ is contained in a hyperplane. □

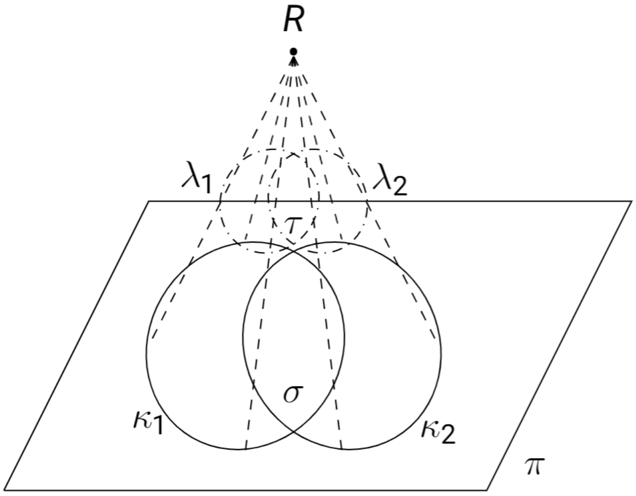

Figure 1: A visualisation of $\mathrm{supp}(c)$ in case there exists a point R and a hyperplane π such that $\mathrm{proj}_{R,\pi}(c) = \alpha_1\kappa_1 + \alpha_2\kappa_2$ for distinct k-subspaces $\kappa_i \subseteq \pi$ and non-zero values $\alpha_i \in \mathbb{F}_p^*$. We define $\lambda_i := \langle R, \kappa_i\rangle$, $\tau := \lambda_1 \cap \lambda_2$ and $\sigma := \kappa_1 \cap \kappa_2$.

Lemma 5.8. *Let c be a codeword of $\mathcal{C}_k(n,q)$ with $q \geqslant 5$ and $\mathrm{wt}(c) \leqslant W(k,q)$, and assume that all codewords of $\mathcal{C}_k(n-1,q)$ with weight at most $W(k,q)$ are linear combinations of at most two k-spaces. Consider a point $R \notin \mathrm{supp}(c)$ and a hyperplane $\pi \not\ni R$; let $\kappa_1, \kappa_2 \in G_k(\pi)$, $\kappa_1 \neq \kappa_2$, and let $\alpha_1, \alpha_2 \in \mathbb{F}_p^*$. Define $\lambda_i := \langle R, \kappa_i\rangle$ and $\tau := \lambda_1 \cap \lambda_2$. Assume that precisely one of the following holds:*

(1) q is even and $\mathrm{proj}_{R,\pi}(c) = \kappa_1$, or

(2) $\mathrm{proj}_{R,\pi}(c) = \alpha_1\kappa_1 + \alpha_2\kappa_2$.

Then there exists a k-space H such that more than $\frac{1}{2}\theta_k$ points of H have the same non-zero value w.r.t. c.

Proof. Remark that, by Lemma 5.2 (2) and (4), the assumptions imply that $\mathrm{proj}_{R',\pi'}(c)$ is a linear combination of at most two k-subspaces of π', for every point $R' \notin \mathrm{supp}(c)$ and every hyperplane $\pi' \not\ni R$.
First, assume that (2) holds.

Observation 1. Every line in $\lambda_1 \setminus \tau$ through R is tangent to $\mathrm{supp}(c)$.

Indeed, take such a line l. We know that $\alpha_1 = \mathrm{proj}_{R,\pi}(c)(l \cap \pi) = c \cdot l$. By Lemma 5.6, l is either a short or a long secant to $\mathrm{supp}(c)$. By the same lemma, l cannot be a 0- or a q-secant, as else $\alpha_1 = 0$. Finally, l cannot be a 2-secant either, as else, by Lemma 5.6 and Lemma 5.2, $\alpha_1 = c \cdot l = c \cdot \mathbf{1} = \mathrm{proj}_{R,\pi}(c) \cdot \mathbf{1} = \alpha_1 + \alpha_2$, which would imply that $\alpha_2 = 0$.

Observation 2. All 2-secants to $\mathrm{supp}(c)$ in λ_1 are contained in τ.

Let s be a 2-secant to $\mathrm{supp}(c)$ in λ_1 that is not contained in τ. Take a point $S \in s \setminus \tau$. By Remark 5.3, we can choose π to be a hyperplane not through R, intersecting s in S. Note that this also means that s intersects κ_1 in S. As $q > 2$, we can choose a point $R_1 \in s \setminus (\mathrm{supp}(c) \cup \tau)$. By Observation 1, as $R_1 \in \lambda_1 \setminus \tau$, RR_1 is tangent to $\mathrm{supp}(c)$ and hence the unique point of $\mathrm{supp}(c)$ on RR_1 must have value α_1. Denote $T = RR_1 \cap \kappa_1$.
In this way, we can see that

- $\mathrm{proj}_{R_1,\pi}(c)(S) = \alpha_1 + \alpha_2$, by Lemma 5.6 and Lemma 5.2 (5), and
- $\mathrm{proj}_{R_1,\pi}(c)(T) = \alpha_1$, implying in particular that $\mathrm{proj}_{R_1,\pi}(c) \neq \mathbf{0}$.

Therefore, $\mathrm{proj}_{R_1,\pi}(c)$ must also be a linear combination of exactly two distinct k-spaces, as else $\mathrm{proj}_{R_1,\pi}(c) = \alpha_1\kappa$ for a certain k-space $\kappa \subseteq \pi$, implying that $\alpha_1 = \mathrm{proj}_{R_1,\pi}(c)\cdot\mathbf{1} = c\cdot\mathbf{1} = \mathrm{proj}_{R,\pi}(c)\cdot\mathbf{1} = \alpha_1+\alpha_2$ by Lemma 5.2 (5), a contradiction.
Furthermore, it's clear that $\mathrm{proj}_{R_1,\pi}(c)$ and $\mathrm{proj}_{R,\pi}(c)$ cannot share the same k-subspaces of π, as else the points $S, T \in \kappa_1 \setminus \tau$ must have the same value w.r.t. $\mathrm{proj}_{R_1,\pi}(c)$, resulting in $\alpha_1 = \alpha_1 + \alpha_2$, a contradiction yet again. Hence, we can find a k-space $\kappa_3 \notin \{\kappa_1, \kappa_2\}$ in π containing, by Observation 1, at least q^k points in a k-dimensional affine subspace, each connected to R_1 by a tangent line to $\mathrm{supp}(c)$.
This means that there are at least $q^k - 2q^{k-1} + \theta_{k-2}$ points of $\mathrm{supp}(c)$ outside of $\lambda_1 \cup \lambda_2$. Hence, we get the following contradiction: $\mathrm{wt}(c) \geqslant |(\lambda_1\cup\lambda_2)\cap\mathrm{supp}(c)| + |\lambda_3 \setminus (\lambda_1\cup\lambda_2)\cap\mathrm{supp}(c)| \geqslant 2q^k + q^k - 2q^{k-1} + \theta_{k-2} = 3q^k - 2q^{k-1} + \theta_{k-2} > W(k,q)$. As a result, Observation 2 is found to be true.

Define $\mathcal{S} := (\lambda_1 \setminus \tau) \cap \mathrm{supp}(c)$. By Lemma 5.6, Observation 2 and Lemma 5.7, there exists a k-space H in λ_1 such that either $\mathcal{S} \subseteq H$ or $\big(\lambda_1 \setminus \mathcal{S}\big) \subseteq H$. The latter would imply that $\mathrm{wt}(c) \geqslant |\lambda_1 \setminus (H \cup \tau)| \geqslant q^{k+1} - q^k > W(k,q)$ as $q \geqslant 5$, a contradiction. Thus, $\mathcal{S} \subseteq H$ must be valid. By Observation 1, all $q^k > \frac{1}{2}\theta_k$ points in $\mathcal{S}$ have non-zero value α_1 w.r.t. c, proving the lemma.

Now assume that (1) holds. The proof stays mainly the same, except for the proof of Observation 4; we will indicate what arguments need to be changed or added in order to keep all proofs valid. In general, every instance of α_1 and α_2 can be replaced by 1, as q is even, and every instance of κ_2 and τ need to be replaced by $\emptyset$. Therefore, Observation 1 becomes the following statement:

Observation 3. Every line in λ_1 through R is tangent to $\mathrm{supp}(c)$.

This can be proven using exactly the same arguments as before: such a line l can only be a tangent line or a 2-secant, and if l is a 2-secant, we would obtain $1 = \alpha_1 = c \cdot l = 1 + 1 = 0$, as q is even, a contradiction.
Observation 2 changes to the following:

Observation 4. There are no 2-secants to $\mathrm{supp}(c)$ contained in λ_1.

We can repeat all notations and arguments used to prove Observation 2 (keeping in mind that τ is replaced by $\emptyset$) and prove that there exists a k-space $\kappa_3 \neq \kappa_1$ in π in which, by Observation 3, each point is connected to R_1 by a tangent line to $\mathrm{supp}(c)$.
Remark that, as q is even, $\mathrm{proj}_{R_1,\pi}(c)(S) = 0$, implying that $S \notin \kappa_3$ as $\mathrm{proj}_{R_1,\pi}(c)(Q) = 1$ for every $Q \in \kappa_3$. Therefore, for each point P of the at least $\theta_k - \theta_{k-1} = q^k$ points of $\mathrm{supp}(c)$ in $\lambda_3 := \langle R_1, \kappa_3\rangle$ not contained in λ_1, the plane $\sigma_P := \langle s, P\rangle$ intersects λ_1 in the 2-secant s and λ_3 in the tangent line R_1P (Observation 3). If $|\sigma_P \cap \mathrm{supp}(c)| \leqslant 4$, then a clever choice of a point $R_2 \in \sigma_P \setminus \mathrm{supp}(c)$ (and a hyperplane $\pi_2 \not\ni R_2$) will result in the existence of a $|\sigma_P \cap \mathrm{supp}(c)|$-secant to $\mathrm{supp}(\mathrm{proj}_{R_2,\pi_2}(c))$, contradicting Lemma 5.6 as $q \geqslant 5$.
In conclusion, for every such point P, we find at least 2 points of $\mathrm{supp}(c)$ outside of $\lambda_1 \cup \lambda_3$ by considering the plane σ_P. As R_1P is tangent to $\mathrm{supp}(c)$, each choice of such a P will result 2 extra points we haven't considered before. Hence, $\mathrm{wt}(c) \geqslant |\lambda_1 \cap \mathrm{supp}(c)| + 3|(\lambda_3\setminus\lambda_1)\cap\mathrm{supp}(c)| \geqslant \theta_k + 3q^k = 4q^k + 3\theta_{k-1} > W(k,q)$, a contradiction.

Given Observation 3 and 4, we can repeat the same arguments as before to conclude the proof. □

Theorem 5.9. *If c is a codeword of $\mathcal{C}_k(n,q)$, with* $\mathrm{wt}(c) \leqslant W(k,q)$*, then c is a linear combination of at most two k-spaces. Moreover, if $q \in Q_3 \cup Q_4 \cup Q_5$, then this bound is tight.*

Proof. The proof will be done by induction on n. The case $n = k+1$ is Result 3.3. So assume that $n \geqslant k+2$ and that the theorem holds for the code $\mathcal{C}_k(n-1,q)$. Assume to the contrary that there exist codewords of $\mathcal{C}_k(n,q)$, with weight at most $W(k,q)$, which can't be written as a linear combination of at most two k-spaces. Let c be such a codeword of smallest possible weight. We will derive a contradiction by making use of the following observation.

Observation 1. There cannot exist a k-space κ such that more than $\frac{1}{2}\theta_k$ points of κ have the same non-zero value α w.r.t. c.

This follows from the fact that if such a k-space κ would exist, then $\mathrm{wt}(c - \alpha\kappa) < \mathrm{wt}(c)$. Since $c - \alpha\kappa \in \mathcal{C}_k(n, q)$, this would mean that $c - \alpha\kappa$ is a linear combination of at most two k-spaces. This is only possible if c is a linear combination of precisely three k-spaces. But then $\mathrm{wt}(c) > W(k, q)$, by Lemma 5.4, a contradiction.

Given a hyperplane π and a point $R \notin \pi \cup \mathrm{supp}(c)$, there are three possibilities for $\mathrm{proj}_{R,\pi}(c)$:

(P0) $\mathrm{proj}_{R,\pi}(c) = \mathbf{0}$,

(P1) $\mathrm{proj}_{R,\pi}(c) = \alpha\kappa$, with $\alpha \in \mathbb{F}_p^*$ and κ a k-space of π, or

(P2) $\mathrm{proj}_{R,\pi}(c) = \alpha_1\kappa_1 + \alpha_2\kappa_2$, with $\alpha_i \in \mathbb{F}_p^*$, and κ_i distinct k-spaces of π.

This follows from the fact that $\mathrm{wt}(\mathrm{proj}_{R,\pi}(c)) \leqslant \mathrm{wt}(c) \leqslant W(k, q)$ (Lemma 5.2 (4)), hence due to the induction hypothesis, $\mathrm{proj}_{R,\pi}(c)$ is characterised as a linear combination of at most two k-spaces.

Case 1: Possibility (P2) never occurs.

Take a point $P \in \mathrm{supp}(c)$, then there exists a tangent line l to $\mathrm{supp}(c)$ through P. Otherwise, each of the θ_{n-1} lines through P contains another point of $\mathrm{supp}(c)$, implying that $\mathrm{wt}(c) > \theta_{n-1} > W(k, q)$, since $n \geqslant k + 2$, a contradiction. Now take a point $R \in l \setminus \{P\}$ and a hyperplane π with $\pi \cap l = \{P\}$. Then $\mathrm{proj}_{R,\pi}(c)(P) = \sum_{Q \in PR} c(Q) = c(P)$. Hence, $\mathrm{proj}_{R,\pi}(c)$ can't be $\mathbf{0}$, which means (P1) is the only possibility that can arise. So $\mathrm{proj}_{R,\pi}(c) = \alpha\kappa$ for some $\alpha \in \mathbb{F}_p^*$ and some k-space κ. It now follows that $\alpha = c(P)$ and $\mathrm{proj}_{R,\pi}(c) \cdot \mathbf{1} = \alpha$, so by Lemma 5.2 (5), $c(P) = c \cdot \mathbf{1}$. Since this holds for all points of $\mathrm{supp}(c)$, they all have the same non-zero value $\alpha := c \cdot \mathbf{1}$ w.r.t. c. Note that this also means that $\mathrm{proj}_{R,\pi}(c) \cdot \mathbf{1}$ can never be zero, which means that possibility (P0) doesn't occur, for any choice of hyperplane π and point $R \notin \pi \cup \mathrm{supp}(c)$.
Remark that, if $q \geqslant 5$ and q is even, Lemma 5.8 can be used to obtain a contradiction to Observation 1. Therefore, we can assume that q is 2, 4 or odd.
Taking an arbitrary hyperplane π and a point $R \notin \pi \cup \mathrm{supp}(c)$, we conclude that $\mathrm{proj}_{R,\pi}(c) = \alpha\kappa$, for some k-space κ in π. Define $\lambda := \langle R, \kappa \rangle$. For every point $P \in \kappa$, the line PR intersects $\mathrm{supp}(c)$. Therefore, the $(k+1)$-space λ intersects $\mathrm{supp}(c)$ in at least θ_k points.

Since $k \leqslant n - 2$, there exists a hyperplane π' through λ. Take a point $R' \notin \pi' \cup \mathrm{supp}(c)$, then $\mathrm{proj}_{R',\pi'}(c) = \alpha\kappa'$ for some k-space κ' in π'. We define the following numbers:

$$x_1 = |\mathrm{supp}(c) \cap \pi'| \geqslant \theta_k, \qquad x_2 = |(\mathrm{supp}(c) \cap \pi') \setminus \kappa'|, \qquad x_3 = |\kappa' \setminus \mathrm{supp}(c)|.$$

If $P \in (\mathrm{supp}(c) \cap \pi') \setminus \kappa'$, then

$$0 = \mathrm{proj}_{R',\pi'}(c)(P) = \sum_{Q \in PR'} c(Q) \equiv \alpha \cdot |\mathrm{supp}(c) \cap PR'| \pmod{p}.$$

Hence, PR' contains $0 \pmod{p}$ points of $\mathrm{supp}(c)$, which means PR' contains at least $p - 1$ points of $\mathrm{supp}(c) \setminus \pi'$. Remark that, if q is odd and $q \neq 3$, then $p > 2$ and we can apply Lemma 5.6 to state that PR' contains at least $q - 1$ points of $\mathrm{supp}(c) \setminus \pi'$. If $P \in \kappa' \setminus \mathrm{supp}(c)$, then PR' contains at least one point of $\mathrm{supp}(c) \setminus \pi'$. This yields

$$\begin{cases} (p-1)x_2 + x_3 \leqslant |\mathrm{supp}(c) \setminus \pi'| = \mathrm{wt}(c) - x_1 \leqslant 2\theta_k - \theta_k = \theta_k & \text{if } q \leqslant 4, \\ (q-1)x_2 + x_3 \leqslant |\mathrm{supp}(c) \setminus \pi'| = \mathrm{wt}(c) - x_1 \leqslant W(k, q) - \theta_k & \text{if } q > 4 \text{ is odd.} \end{cases} \tag{1}$$

Also note that $|\kappa' \cap \mathrm{supp}(c)| = x_1 - x_2$ and $x_3 = |\kappa'| - |\kappa' \cap \mathrm{supp}(c)| = \theta_k - x_1 + x_2$. Hence the system of equations (1) becomes

$$\begin{cases} (p-1)x_2 + \theta_k - x_1 + x_2 \leqslant \theta_k & \text{if } q \leqslant 4, \\ (q-1)x_2 + \theta_k - x_1 + x_2 \leqslant 3q^k - 2q^{k-1} + \theta_{k-2} - 1 - \theta_k & \text{if } q > 4 \text{ is odd,} \end{cases}$$

which implies

$$x_2 \leqslant \begin{cases} \frac{x_1}{p} & \text{if } q \leqslant 4, \\ \frac{x_1}{q} + q^{k-1} & \text{if } q > 4 \text{ is odd,} \end{cases}$$

Thus, if $q \leqslant 4$, we get

$$|\mathrm{supp}(c) \cap \kappa'| = x_1 - x_2 \geqslant \frac{p-1}{p} x_1 \geqslant \frac{p-1}{p}\theta_k. \tag{2}$$

If $p = 2$, then θ_k is odd, hence $|\mathrm{supp}(c) \cap \kappa'| > \frac{1}{2}\theta_k$ since the left-hand side must be an integer. Otherwise, $q = p = 3$ and $\frac{p-1}{p} = \frac{2}{3}$, which also implies $|\mathrm{supp}(c) \cap \kappa'| > \frac{1}{2}\theta_k$. This yields a contradiction to Observation 1, since all points of $\mathrm{supp}(c)$ have the same value w.r.t. c.
If $q > 4$ is odd, we get the following variant of equation (2).

$$|\mathrm{supp}(c) \cap \kappa'| = x_1 - x_2 \geqslant \frac{q-1}{q}\theta_k - q^{k-1} > \frac{1}{2}\theta_k.$$

The last inequality holds as $q > 4$. This results yet again in a contradiction to Observation 1.

Case 2: Possibility (P2) does occur.
If $q \geqslant 5$, Lemma 5.8 implies a contradiction to Observation 1. Therefore, we can assume that $q \leqslant 4$, which implies that $W(k, q) = 2q^k$.
Take a hyperplane π and a point $R \notin \pi \cup \mathrm{supp}(c)$ such that $\mathrm{proj}_{R,\pi}(c) = \alpha_1\kappa_1 + \alpha_2\kappa_2$ for some $\alpha_i \in \mathbb{F}_p^*$ and distinct k-spaces κ_i of π. Define the following notation (see Figure 1 accompanying Lemma 5.8):

$$\sigma := \kappa_1 \cap \kappa_2, \qquad s := \dim(\sigma), \qquad \tau := \langle R, \sigma \rangle, \qquad \lambda_i := \langle R, \kappa_i \rangle.$$

First, remark that $\mathrm{supp}(c) \subseteq \lambda_1 \cup \lambda_2$. Indeed, as $\mathrm{wt}(c) \leqslant 2q^k$ and $s \leqslant k-1$, we know that $\lambda_1 \cup \lambda_2$ contains at least $2(\theta_k - \theta_{k-1}) = 2q^k$ points of $\mathrm{supp}(c)$. This is only possible if $\mathrm{wt}(c) = 2q^k$ and thus $\mathrm{supp}(c) \subseteq \lambda_1 \cup \lambda_2$. Note that this means that $\mathrm{proj}_{R,\pi}(c) = \alpha_1(\kappa_1 - \kappa_2)$, and $s = k - 1$.
Now take a point $Q \in \lambda_1 \setminus (\lambda_2 \cup \mathrm{supp}(c))$. We can assume, w.l.o.g., that $Q \notin \pi$ (otherwise, by Remark 5.3, we choose another hyperplane π). Then Q projects every point of λ_1 onto a point of $\kappa_1 \subseteq \pi$, and for every point P of $\lambda_2 \setminus \tau$, QP either intersects $\mathrm{supp}(c)$ in P or doesn't intersect $\mathrm{supp}(c)$ at all. Hence, the points of $(\lambda_2 \setminus \tau) \cap \mathrm{supp}(c)$ are projected by Q onto points with non-zero value w.r.t. $\mathrm{proj}_{Q,\pi}(c)$. In particular, $\mathrm{proj}_{Q,\pi}(c) \neq \mathbf{0}$. By Lemma 5.2 (5), this implies that $\mathrm{proj}_{Q,\pi}(c)$ is a linear combination of precisely two k-spaces. Furthermore, as $\mathrm{wt}(c) = 2q^k$, we know that $\mathrm{proj}_{Q,\pi}(c)$ is the difference of two distinct k-spaces through a $(k-1)$-space.
The fact that $\mathrm{wt}(\mathrm{proj}_{Q,\pi}(c)) = 2q^k$ is only possible if no line through Q contains more than one point of $\mathrm{supp}(c)$. In this way, we see that all points of $\kappa_1 \setminus \sigma$ must have value α_1 w.r.t. $\mathrm{proj}_{Q,\pi}(c)$. Thus, $\mathrm{proj}_{Q,\pi}(c) = \alpha_1(\kappa_1 - \rho)$ for some k-space ρ in π.[1] This means that all points of $\mathrm{supp}(c) \cap (\lambda_2 \setminus \tau)$ have value $-\alpha_1$ and lie in the space $\mu := \lambda_2 \cap \langle Q, \rho \rangle$. Note that $\dim(\mu) \leqslant k$ and μ contains $q^k > \frac{1}{2}\theta_k$ points of $\mathrm{supp}(c)$ with value $-\alpha_1$ w.r.t. c. Observation 1 yields the desired contradiction.

[1] Beware that if $q = 2$ and $c = \kappa_1 + \kappa_2$, with κ_1 and κ_2 k-spaces through a $(k-1)$-space, these spaces κ_1 and κ_2 are not uniquely determined by c. This is because, if $K = \langle \kappa_1, \kappa_2 \rangle$, then $K \setminus \mathrm{supp}(c)$ is a k-space κ_3. If κ_1' and κ_2' are distinct k-spaces in K, intersecting κ_3 in the same $(k-1)$-space, then also $c = \kappa_1' + \kappa_2'$.

If $q \in Q_3 \cup Q_4 \cup Q_5$, then the bound is tight because it is tight for $\mathcal{C}_k(k+1, q)$ (see Result 3.3) and we can interpret $\mathcal{C}_k(k+1, q)$ as a subcode of $\mathcal{C}_k(n, q)$ by restricting the generating set $G_k^{(0)}(n, q)$ of $\mathcal{C}_k(n, q)$ to $G_k^{(0)}(\Pi)$ for some $(k+1)$-space Π in $\mathrm{PG}(n, q)$. This way we see that $\mathcal{C}_k(n, q)$ must also contain codewords of weight $W(k, q) + 1$. Note that $W(k, q) + 1$ exceeds $2\theta_k$, which is an upper bound on the weight of a linear combination of two k-spaces. □

Corollary 5.10. *If c is a codeword of $\mathcal{H}_{0,k}(n, q)$, with $\mathrm{wt}(c) \leqslant W(k, q)$, then c is a scalar multiple of the difference of two k-spaces. In particular, the minimum weight of $\mathcal{H}_{0,k}(n, q)$ is $2q^k$, and the minimum weight codewords are scalar multiples of the difference of two k-spaces through a common $(k-1)$-subspace.*

Proof. The arguments are the same as in Step 3 of the proof of Theorem 6.7. □

Remark 5.11. It is not difficult to write down the weight spectrum of $\mathcal{C}_k(n, q)$ explicitly for weights up to $W(k, q)$. For all q, the minimum weight codewords have weight θ_k and are the scalar multiples of k-spaces. The next weight is $2q^k$ and is attained only by the scalar multiples of the difference of two k-spaces intersecting in a $(k-1)$-space. In general, if $\alpha_1, \alpha_2 \in \mathbb{F}_p^*$ and $\kappa_1, \kappa_2 \in G_k$ with $\kappa_1 \neq \kappa_2$, then $\mathrm{wt}(\alpha_1\kappa_1 + \alpha_2\kappa_2) = 2\theta_k - (1+\varepsilon)\theta_{\dim(\kappa_1 \cap \kappa_2)}$, with $\varepsilon = 1$ if $\alpha_1 = -\alpha_2$, and $\varepsilon = 0$ otherwise.
In particular, we know that $[2\theta_k - \theta_{2k-n} + 1, W(k, q)]$ is a gap in the weight spectrum. This interval in non-empty if $q \notin Q_1$ and if either $q \notin Q_2$ or $2k \geqslant n$.

6 Codes of j- and k-spaces

The main goal of this section is generalising Theorem 5.9 to all codes $\mathcal{C}_{j,k}(n, q)$. The following map, which is essentially due to Bagchi & Inamdar [BI02], will prove to be very helpful.[2]

Definition 6.1. Looking at $V(j, n, q)$, the elements of $G_j^{(j)}$ form the standard basis. Given an i-space ι of $\mathrm{PG}(n, q)$, with $-1 \leqslant i < j$, we take an $(n-i-1)$-space π of $\mathrm{PG}(n, q)$, skew to ι. Consider the unique linear map $\text{प}_\iota : V(j, n, q) \to V(j-i-1, \pi)$ satisfying, for all $\lambda \in G_j^{(j)}$,

$$\text{प}_\iota(\lambda) = \begin{cases} \lambda \cap \pi & \text{if } \iota \subset \lambda, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

This means that, given $v \in V(j, n, q)$ and a $(j-i-1)$-space $\mu \subset \pi$, we have $\text{प}_\iota(v)(\mu) = v(\langle \mu, \iota \rangle)$.

Note that प_ι is closely related to taking the quotient of $\mathrm{PG}(n, q)$ through the space ι. The choice of π doesn't make a (qualitative) difference for the definition of प_ι.

Lemma 6.2 ([BI02, Theorem 1])**.** *Assume that $c \in \mathcal{C}_{j,k}(n, q)$, with $j \geqslant 1$, and let ι be an i-space of $\mathrm{PG}(n, q)$, with $-1 \leqslant i < j$. Then $\text{प}_\iota(c) \in \mathcal{C}_{j-i-1,k-i-1}(n-i-1, q)$.*

Proof. Take a $\kappa \in G_k^{(j)}$. It is easy to see that

$$\text{प}_\iota(\kappa) = \begin{cases} \kappa \cap \pi & \text{if } \iota \subset \kappa, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

which implies that the image of $G_k(n, q)^{(j)}$ under प_ι is $G_{k-i-1}(\pi)^{(j)} \cup \{\mathbf{0}\}$. These sets generate $\mathcal{C}_{j,k}(n, q)$ and $\mathcal{C}_{j-i-1,k-i-1}(n-i-1, q)$, respectively. Hence, it follows that $\text{प}_\iota\big(\mathcal{C}_{j,k}(n, q)\big) = \mathcal{C}_{j-i-1,k-i-1}(n-i-1, q)$. □

[2] In this section, we will denote two distinct projections with Devanagari symbols. These can be imported in LaTeX using the package `devanagari`. In Definition 6.1, we introduce the symbol प (pronounced 'pa' with corresponding command `{\dn p}`), while, in Definition 6.3, we use the symbol ल (pronounced 'la' with corresponding command `{\dn l}`).

Another map that will serve as a useful tool is the following.

Definition 6.3. Take an integer i, with $0 \leqslant i < j$. For each $v \in V(j,n,q)$ we define $\text{ल}_i(v) \in V(i,n,q)$ as

$$\text{ल}_i(v) : \iota \mapsto \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} v(\lambda).$$

This means that the value of an i-space ι w.r.t. $\text{ल}_i(v)$ is the sum of the values w.r.t. v of all j-spaces λ through ι. We can view $\text{ल}_i : v \mapsto \text{ल}_i(v)$ as a mapping from $V(j,n,q)$ to $V(i,n,q)$. We will denote ल_0 by ल.

Lemma 6.4. *The map ल_i is linear and $\text{ल}_i\big(\mathcal{C}_{j,k}(n,q)\big) = \mathcal{C}_{i,k}(n,q)$.*

Proof. Take $\alpha, \beta \in \mathbb{F}_p$ and $v, w \in V(j,n,q)$. Let ι be an i-space of $\mathrm{PG}(n,q)$. Then

$$\begin{aligned}
\text{ल}_i(\alpha v + \beta w)(\iota) &= \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} (\alpha v + \beta w)(\lambda) = \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} (\alpha v(\lambda) + \beta w(\lambda)) \\
&= \alpha \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} v(\lambda) + \beta \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} w(\lambda)) = \alpha\, \text{ल}_i(v)(\iota) + \beta\, \text{ल}_i(w)(\iota).
\end{aligned}$$

Since this holds for every i-space ι, $\text{ल}_i(\alpha v + \beta w) = \alpha\, \text{ल}_i(v) + \beta\, \text{ल}_i(w)$.
Now take a k-space κ and an i-space ι.

$$\text{ल}_i(\kappa^{(j)})(\iota) = \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} \kappa^{(j)}(\lambda) = \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda \subset \kappa}} 1 = \begin{cases} \begin{bmatrix} k-i \\ j-i \end{bmatrix}_q \equiv 1 \pmod p & \text{if } \iota \subset \kappa, \\ 0 & \text{otherwise,} \end{cases} = \kappa^{(i)}(\iota).$$

This means that $\text{ल}_i(\kappa^{(j)}) = \kappa^{(i)}$. Hence, the generators of $\mathcal{C}_{j,k}(n,q)$ are mapped to the generators of $\mathcal{C}_{i,k}(n,q)$. Since ल_i is linear, this proves that $\text{ल}_i\big(\mathcal{C}_{j,k}(n,q)\big) = \mathcal{C}_{i,k}(n,q)$. $\square$

Lemma 6.5. *Assume that $v \in V(j,n,q)$ and $0 \leqslant i < j$. Then $\text{ल}\big(\text{ल}_i(v)\big) = \text{ल}(v)$.*

Proof. Take an arbitrary point P in $\mathrm{PG}(n,q)$. We need to prove that $\text{ल}\big(\text{ल}_i(v)\big)(P) = \text{ल}(v)(P)$.

$$\begin{aligned}
\text{ल}\big(\text{ल}_i(v)\big)(P) &= \sum_{\substack{\iota \in G_i \\ P \in \iota}} \text{ल}_i(v)(\iota) = \sum_{\substack{\iota \in G_i \\ P \in \iota}} \sum_{\substack{\lambda \in G_j \\ \iota \subset \lambda}} v(\lambda) = \sum_{\substack{\lambda \in G_j \\ P \in \lambda}} v(\lambda) \Big(\sum_{\substack{\iota \in G_i \\ P \in \iota \subset \lambda}} 1 \Big) \\
&= \sum_{\substack{\lambda \in G_j \\ P \in \lambda}} v(\lambda) \begin{bmatrix} j \\ i \end{bmatrix}_q \equiv \sum_{\substack{\lambda \in G_j \\ P \in \lambda}} v(\lambda) = \text{ल}(v)(P) \pmod p. \qquad \square
\end{aligned}$$

The following lemma shows the interaction between प and ल.

Lemma 6.6. *Assume that $c \in \mathcal{C}_{j,k}(n,q)$, and let ι be an i-space, with $0 \leqslant i < j$. Then $\text{ल}_i(c)(\iota) = \text{प}_\iota(c) \cdot \mathbf{1}$. Hence, $\text{ल}_i(c)(\iota) = 0$ if and only if $\text{प}_\iota(c) \in \mathcal{H}_{j-i-1,k-i-1}(n-i-1,q)$.*

Proof. It is easy to see that both $\text{ल}_i(c)(\iota)$ and $\text{प}_\iota(c) \cdot \mathbf{1}$ equal the sum of the values w.r.t. c of all j-spaces through ι. We know that $\text{प}_\iota(c) \in \mathcal{C}_{j-i-1,k-i-1}(n-i-1,q)$. By Lemma 4.1 (2), this means that $\text{प}_\iota(c) \in \mathcal{H}_{j-i-1,k-i-1}(n-i-1,q)$ if and only if $\text{प}_\iota(c) \cdot \mathbf{1} = 0$. $\square$

We can now characterise all codewords of $\mathcal{C}_{j,k}(n,q)$ up to weight $W(j,k,q)$. If q is large enough, then this bound exceeds $2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q$, which is at least the maximum weight of a linear combination of two k-spaces (with equality if and only if $n > 2k - j$).

Theorem 6.7. *(1) If c is a codeword of $\mathcal{C}_{j,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(j,k,q)$, then c is a linear combination of at most two k-spaces.*

(2) If c is a codeword of $\mathcal{H}_{j,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(j,k,q)$, then c is a scalar multiple of the difference of two k-spaces. In particular, if $q \notin Q_1$, then the minimum weight of $\mathcal{H}_{j,k}(n,q)$ is $2q^{k-j}\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]_q$, and the minimum weight codewords are scalar multiples of the difference of two k-spaces through a common $(k-1)$-space.

Proof. We refer to Theorem 6.8 for the case $q \in Q_1$. Hence, throughout the proof, we will assume that $q \notin Q_1$.

We will prove this by induction on j. If $j = 0$, this follows from Theorem 5.9 and Corollary 5.10, as $W(0,k,q) \leqslant W(k,q)$. So assume that $j \geqslant 1$ and that the theorem holds for all codes $\mathcal{C}_{j',k'}(n',q)$, with $j' < j$, and $j' < k' < n'$.

Step 1: Attain a lower bound on the minimum weight of $\ker(\text{ल}_{j-1}) \cap \mathcal{C}_{j,k}(n,q)$.

Let c be a non-zero codeword of $\mathcal{C}_{j,k}(n,q)$, with $\text{ल}_{j-1}(c) = \mathbf{0}$. We will find a lower bound on $\mathrm{wt}(c)$ by performing a double count on the set

$$S := \{(P,\lambda) : P \in \mathrm{supp}_0(c),\ P \in \lambda \in \mathrm{supp}(c)\}.$$

We know that $c \neq \mathbf{0}$ means that $\mathrm{supp}(c) \neq \emptyset$, hence $\mathrm{supp}_{j-1}(c) \neq \emptyset$. Take a subspace $\iota \in \mathrm{supp}_{j-1}(c)$. It follows from Lemma 6.6 that $\text{प}_\iota(c) \in \mathcal{H}_{0,k-j}(n-j,q)$. Recall that $\mathrm{wt}(\text{प}_\iota(c))$ equals the number of j-spaces of $\mathrm{supp}(c)$ through ι. Since $\iota \in \mathrm{supp}_{j-1}(c)$, this number is not zero. Therefore, $\text{प}_\iota(c)$ is a non-zero codeword of $\mathcal{H}_{0,k-j}(n-j,q)$. Thus, by Corollary 5.10, we have that $\mathrm{wt}(\text{प}_\iota(c)) \geqslant 2q^{k-j}$. Hence, $\mathrm{supp}(c)$ contains at least $2q^{k-j}$ j-spaces through ι. This yields that

$$|\mathrm{supp}_0(c)| \geqslant \theta_{j-1} + 2q^{k-j}(\theta_j - \theta_{j-1}) > 2q^k.$$

Now take a point $P \in \mathrm{supp}_0(c)$. On the one hand, Lemma 6.5 assures us that $\text{ल}(c)(P) = \text{ल}(\text{ल}_{j-1}(c))(P) = \text{ल}(\mathbf{0})(P) = 0$. Lemma 6.6 then implies that $\text{प}_P(c) \in \mathcal{H}_{j-1,k-1}(n-1,q)$. On the other hand, $P \in \mathrm{supp}_0(c)$, so $\text{प}_P(c) \neq \mathbf{0}$. Using the induction hypothesis, we get $\mathrm{wt}(\text{प}_P(c)) \geqslant 2q^{k-j}\left[\begin{smallmatrix}k-1\\j-1\end{smallmatrix}\right]_q$. Thus, the number of j-spaces of $\mathrm{supp}(c)$ through P is at least $2q^{k-j}\left[\begin{smallmatrix}k-1\\j-1\end{smallmatrix}\right]_q$. This yields that

$$\mathrm{wt}(c)\theta_j = |S| \geqslant |\mathrm{supp}_0(c)| \cdot 2q^{k-j}\begin{bmatrix}k-1\\j-1\end{bmatrix}_q > 4q^{2k-j}\begin{bmatrix}k-1\\j-1\end{bmatrix}_q.$$

One can check that

$$\frac{q^k}{\theta_j} > \left(1-\frac{1}{q}\right)\frac{q^{k+1}-1}{q^{j+1}-1} \qquad \text{and} \qquad q^{k-j} > \left(1-\frac{1}{q}\right)\frac{q^k-1}{q^j-1}.$$

Therefore, if we take into account that $q \geqslant 11$, the above inequalities imply that

$$\mathrm{wt}(c) > 4\frac{q^k}{\theta_j}q^{k-j}\begin{bmatrix}k-1\\j-1\end{bmatrix}_q > 4\left(1-\frac{1}{11}\right)^2\frac{q^{k+1}-1}{q^{j+1}-1}\frac{q^k-1}{q^j-1}\begin{bmatrix}k-1\\j-1\end{bmatrix}_q > 3.3\begin{bmatrix}k+1\\j+1\end{bmatrix}_q$$

Note that, in particular, $\mathrm{wt}(c) > W(j,k,q)$.

Step 2: Applying this lower bound to characterise low weight codewords.

Assume that c is a codeword of $\mathcal{C}_{j,k}(n,q)$, with $\mathrm{wt}(c) \leqslant W(j,k,q)$. Now, double count the set

$$S := \left\{(\iota,\lambda) \,:\, \iota \in \mathrm{supp}_{j-1}(c),\ \iota \subset \lambda \in \mathrm{supp}(c)\right\}.$$

We know that if $\iota \in \operatorname{supp}_{j-1}(c)$, then $\text{प}_\iota(c)$ is a non-zero codeword of $\mathcal{C}_{0,k-j}(n-j,q)$. Therefore, $\operatorname{wt}(\text{प}_\iota(c)) \geqslant \theta_{k-j}$. Note that $\operatorname{wt}(\text{प}_\iota(c))$ equals the number of j-spaces $\lambda \in \operatorname{supp}(c)$ through ι. Also note that $\operatorname{supp}(\text{ल}_{j-1}(c)) \subseteq \operatorname{supp}_{j-1}(c)$. This yields

$$\operatorname{wt}(c)\theta_j = |S| = \sum_{\iota \in \operatorname{supp}_{j-1}(c)} \operatorname{wt}(\text{प}_\iota(c)) \geqslant \operatorname{wt}(\text{ल}_{j-1}(c))\theta_{k-j}.$$

This means that

$$\operatorname{wt}(\text{ल}_{j-1}(c)) \leqslant \frac{\theta_j}{\theta_{k-j}}\operatorname{wt}(c) \leqslant \frac{\theta_j}{\theta_{k-j}} W(j,k,q) = W(j-1,k,q).$$

The last inequality relies on the fact that $\frac{\theta_j}{\theta_{k-j}} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q = \begin{bmatrix} k+1 \\ j \end{bmatrix}_q$.
The induction hypothesis tells us that $\text{ल}_{j-1}(c)$ is a linear combination of at most two k-spaces. Thus, $\text{ल}_{j-1}(c) = \alpha\kappa_1^{(j-1)} + \beta\kappa_2^{(j-1)}$, for some $\alpha, \beta \in \mathbb{F}_p$, and $\kappa_i \in G_k$. Note that α or β can be zero.
Now assume that $c \neq \alpha\kappa_1^{(j)} + \beta\kappa_2^{(j)}$. If $\operatorname{supp}(c) \subseteq G_j(\kappa_1) \cup G_j(\kappa_2)$, then $\operatorname{supp}(c - \alpha\kappa_1^{(j)} - \beta\kappa_2^{(j)}) \subseteq G_j(\kappa_1) \cup G_j(\kappa_2)$, which would mean that $c - \alpha\kappa_1 - \beta\kappa_2$ were a non-zero codeword of $\ker(\text{ल}_{j-1}) \cap \mathcal{C}_{j,k}(n,q)$ of weight at most $2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q$, contradicting Step 1.
Therefore, there exists a j-space $\lambda \in \operatorname{supp}(c)$, with $\lambda \not\subset \kappa_1 \cup \kappa_2$. Hence, we can choose a $(j-1)$-space $\iota \subset \lambda$, which is not entirely contained in $\kappa_1 \cup \kappa_2$. This means that $\text{ल}_{j-1}(c)(\iota) = \alpha\kappa_1(\iota) + \beta\kappa_2(\iota) = 0$. Since $\iota \in \operatorname{supp}_{j-1}(c)$, this implies $\operatorname{wt}(\text{प}_\iota(c)) \geqslant 2q^{k-j}$. Hence, we find at least $2q^{k-j}$ j-spaces of $\operatorname{supp}(c)$ through ι. Note that all these j-spaces contain at least $\theta_j - 3\theta_{j-1} = q^j - 2\theta_{j-1}$ points P outside of ι, κ_1 and κ_2. Every such point P lies in a unique j-space through ι, hence there at least $2q^{k-j}(q^j - 2\theta_{j-1})$ points in $\operatorname{supp}_0(c)$, outside of $\kappa_1 \cup \kappa_2$. Since these points have value zero w.r.t. $\text{ल}(c)$, they lie in at least $2q^{k-j}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q$ j-spaces of $\operatorname{supp}(c)$. As in Step 1, we obtain

$$\operatorname{wt}(c)\theta_j \geqslant 2q^{k-j} \underbrace{(q^j - 2\theta_{j-1})}_{> q^j \frac{q-3}{q-1}} 2q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q > 4q^{2k-j} \frac{q-3}{q-1} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q.$$

Therefore,

$$\operatorname{wt}(c) \geqslant 4\left(1 - \frac{1}{q}\right)^2 \frac{q-3}{q-1} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q > \left(4 - \frac{16}{q}\right) \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q > W(j,k,q),$$

a contradiction. Hence, $c = \alpha\kappa_1^{(j)} + \beta\kappa_2^{(j)}$.

Step 3: The minimum weight of $\mathcal{H}_{j,k}(n,q)$.
The previous characterisation teaches us that the only codewords of $\mathcal{H}_{j,k}(n,q)$ of weight at most $W(j,k,q) \geqslant 2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q$ are linear combinations of at most two k-spaces. Take such a non-zero codeword $c = \alpha\kappa_1 + \beta\kappa_2$. Then $\alpha + \beta = c \cdot \mathbf{1} = 0$, due to Lemma 4.1 (2). Since α and β can't both be zero (then c would be $\mathbf{0}$), neither of them can be zero. Write $s = \dim(\kappa_1 \cap \kappa_2)$, then $\operatorname{wt}(c) = 2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q - 2\begin{bmatrix} s+1 \\ j+1 \end{bmatrix}_q$. This is minimal if s is maximal. Since κ_1 and κ_2 can't coincide (else c would be $\mathbf{0}$), the maximal value of s is $k-1$. This yields as minimum weight of $\mathcal{H}_{j,k}(n,q)$

$$2\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q - 2\begin{bmatrix} k \\ j+1 \end{bmatrix}_q = 2q^{k-j}\begin{bmatrix} k \\ j \end{bmatrix}_q,$$

and as minimum weight codewords the scalar multiples of the difference of two distinct k-spaces through a $(k-1)$-space. $\square$

We now deal with the case $q \in Q_1$, but formulate the result more generally. This only requires a small modification of the previous proof.

Theorem 6.8. *If c is a codeword of $\mathcal{C}_{j,k}(n,q)$, with*

$$\mathrm{wt}(c) \leqslant \frac{2q^k}{\theta_j}\begin{bmatrix}k\\j\end{bmatrix}_q,$$

then $c = \alpha\kappa$, for some $\alpha \in \mathbb{F}_p$, and $\kappa \in G_k$. As a consequence, the minimum weight of $\mathcal{H}_{j,k}(n,q)$ is larger than $2q^k\begin{bmatrix}k\\j\end{bmatrix}_q/\theta_j$.

Proof. The arguments are essentially the same as the ones used in the proof of Theorem 6.7, so we'll be brief. Assume that c is a non-zero codeword of $\mathcal{C}_{j,k}(n,q)$ with $\mathrm{wt}(c) \leqslant \frac{2q^k}{\theta_j}\begin{bmatrix}k\\j\end{bmatrix}_q$ and the theorem holds for all smaller values of j.
Step 1: Assume that $\text{प}_{j-1}(c) = \mathbf{0}$. Double count the set S as in Step 1 above. We obtain $\mathrm{wt}(c) \geqslant \frac{2q^k+\theta_{j-1}}{\theta_j}\frac{2q^{k-1}}{\theta_{j-1}}\begin{bmatrix}k-1\\j-1\end{bmatrix}_q > \frac{2q^k}{\theta_j}\begin{bmatrix}k\\j\end{bmatrix}_q$, a contradiction.
Step 2: Here we have, similar to the above proof,

$$\mathrm{wt}(\text{प}_{j-1}(c)) \leqslant \frac{\theta_j}{\theta_{k-j}}\mathrm{wt}(c) \leqslant \frac{\theta_j}{\theta_{k-j}}\frac{2q^k}{\theta_j}\begin{bmatrix}k\\j\end{bmatrix}_q = \frac{2q^k}{\theta_{k-j}}\frac{\theta_{k-j}}{\theta_{j-1}}\begin{bmatrix}k\\j-1\end{bmatrix}_q = \frac{2q^k}{\theta_{j-1}}\begin{bmatrix}k\\j-1\end{bmatrix}_q.$$

Therefore, the induction hypothesis implies that $\text{ल}_{j-1}(c) = \alpha\kappa$ for some scalar $\alpha \in \mathbb{F}_p^*$ and a k-space κ. As above, if $c \neq \alpha\kappa$, then $\mathrm{supp}(c) \not\subseteq G_j(\kappa)$. Thus, there exists a $(j-1)$-space $\iota \in \mathrm{supp}_{j-1}(c)$ with $\text{ल}_{j-1}(\iota) = 0$. Then $\text{प}_\iota(c)$ is a non-zero codeword of $\mathcal{H}_{k-j}(n-j,q)$ and we know that $\mathrm{supp}_0(c) \geqslant 2q^k + \theta_{j-1}$. Hence, $\mathrm{wt}(c)\theta_j \geqslant (2q^k + \theta_{j-1})\begin{bmatrix}k\\j\end{bmatrix}_q$, a contradiction.
Step 3: No scalar multiple of a k-space is a non-zero codeword of $\mathcal{H}_{j,k}(n,q)$. □

The minimum weight of $\mathcal{H}_{j,k}(n,q)$ has been an open problem for some time [LSVdV10, Open Problem 4.18]. We have solved this problem for $j = 0$ in Theorem 5.9 and for general j and sufficiently large q in Theorem 6.7.
The authors expect that Theorem 6.7 (2) holds for all values of q. For instance, Theorem 6.7 (1) can be proven to hold for $\mathcal{C}_{1,2}(n,q)$, $q \neq 2$ up to weight $2\theta_2$, which proves (2) for $\mathcal{H}_{1,2}(n,q)$, $q \neq 2$.

As we have done in Remark 5.11, one can now study the weight spectrum of $\mathcal{C}_{j,k}(n,q)$ up to weight $W(j,k,q)$ using Theorem 6.7 and 6.8.

The cyclicity of $\mathcal{C}_{j,k}(n,q)$

A natural question to ask is whether the codes $\mathcal{C}_{j,k}(n,q)$ are cyclic. A code C, where the codewords are denoted as vectors, is *cyclic* if for each codeword $(c_1,\dots,c_n) \in C$, its *right shift* $(c_n, c_1, c_2, \dots, c_{n-1})$ is also a codeword of C.
It has been known for a long time that the codes $\mathcal{C}_k(n,q)$ are cyclic, see e.g. [DGM70]. Denote $g := \begin{bmatrix}n+1\\j+1\end{bmatrix}_q$. Then $\mathcal{C}_{j,k}(n,q)$ is equivalent to a cyclic code if and only if the following holds: there exists some ordering on the j-spaces of $\mathrm{PG}(n,q)$ (write $G_j(n,q) = \{\lambda_1, \lambda_2, \dots, \lambda_g\}$ and let λ_0 be equal to λ_g) such that if $c \in \mathcal{C}_{j,k}(n,q)$, then $R(c) \in \mathcal{C}_{j,k}(n,q)$ as well, with $R(c)(\lambda_i) := c(\lambda_{i-1})$.
Given a k-space κ, this would mean that $R(\kappa)$ is also a codeword of $\mathcal{C}_{j,k}(n,q)$. Furthermore, it's easy to see that $\mathrm{wt}(R(\kappa)) = \mathrm{wt}(\kappa) = \begin{bmatrix}k+1\\j+1\end{bmatrix}_q$, and that $R(\kappa)$ only takes the values 0 and 1. By Result 3.1, this means that $R(\kappa) = \kappa'$ for some k-space κ'.
This means that the map $f : G_j \to G_j : \lambda_i \mapsto \lambda_{i-1}$ maps the j-spaces in a certain k-space to the j-spaces of another k-space. But then f can be extended to a collineation on all subspaces

of $\mathrm{PG}(n,q)$. Note that f works cyclically on the j-spaces, meaning that the permutation group generated by f has a unique orbit when viewed as permutation group of G_j.
Conversely, if such a collineation f exists, we can choose a $\lambda \in G_j$ and write $\lambda_1 = \lambda$, and $\lambda_{i+1} = f(\lambda_i)$. Under this ordering of the j-spaces, $\mathcal{C}_{j,k}(n,q)$ is cyclic. This yields the following statement:

> *Observation* 1. The code $\mathcal{C}_{j,k}(n,q)$ is equivalent to a cyclic code if and only if there exists a collineation f of $\mathrm{PG}(n,q)$, working cyclically on the j-spaces.

It is folklore under finite geometers that the collineations with largest order are Singer cycles, which act cyclically on the points and hyperplanes. However, a reference is hard to find. We will use a similar (but in this context weaker) result that suits our purpose.

Result 6.9 ([Dar05, Corollary 2])**.** *The maximal order of an element of* $\mathrm{GL}(n,q)$ *is* $q^n - 1$.

This leads to the following Theorem.

Theorem 6.10. *The code $\mathcal{C}_{j,k}(n,q)$ is equivalent to a cyclic code if and only if $j = 0$.*

Proof. In the codes we consider, we have the restriction $0 \leqslant j < k < n$. By Observation 1, we need to prove that some collineations work cyclically on the points, but no collineation works cyclically on the j-spaces if $0 < j < n-1$. It is known that Singer cycles are collineations working cyclically on the points and hyperplanes of $\mathrm{PG}(n,q)$, and that such collineations exist for any Desarguesian projective space. Hence, this proves that $\mathcal{C}_k(n,q)$ is equivalent to a cyclic code.
Now assume that $1 \leqslant j \leqslant n-2$. Let f be a collineation on $\mathrm{PG}(n,q)$. The Fundamental Theorem of projective geometry teaches us that $f \in \mathrm{P\Gamma L}(n+1,q)$. This is a quotient group of $\Gamma\mathrm{L}(n+1,q)$, which is a subgroup of $\mathrm{GL}((n+1)h,p)$. Therefore, the order of f cannot exceed the maximal order of an element of $\mathrm{GL}((n+1)h,p)$, which is $p^{(n+1)h} - 1 = q^{n+1} - 1$, by Result 6.9. But if f would work cyclically on the j-spaces of $\mathrm{PG}(n,q)$, then its order would be a multiple of $\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_q$, which exceeds $q^{n+1} - 1$ if $n \geqslant 3$ and $1 \leqslant j \leqslant n-2$. This contradiction concludes the proof. $\square$

7 Minimum weight of the dual code

Throughout [ADSW20] and Section 5 and 6, we characterise small weight codewords of $\mathcal{C}_{j,k}(n,q)$ by starting from $\mathcal{C}_{0,1}(2,q)$ and using induction to generalise the results. Unfortunately, it is not possible to do something similar for the dual code. The problem of determining the minimum weight of $\mathcal{C}_{0,1}(2,q)^\perp$ and characterising its minimum weight codewords is still open in general. However, we can work in the opposite direction, and reduce the minimum weight problem of $\mathcal{C}_{j,k}(n,q)^\perp$ to the codes $\mathcal{C}_{0,1}(n,q)^\perp$. A construction by Bagchi & Inamdar is key.

Construction 7.1 ([BI02, Lemma 4])**.** Consider the code $\mathcal{C}_{j,k}(n,q)^\perp$. Take a $(j-1)$-space ι, and an $(n-j)$-space π, skew to ι. Let c be a codeword of $\mathcal{C}_{k-j}(\pi)^\perp$. Define $c_\iota^+ \in V(j,n,q)$ as

$$c_\iota^+(\lambda) := \begin{cases} c(\lambda \cap \pi) & \text{if } \iota \subset \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then $c_\iota^+ \in \mathcal{C}_{j,k}(n,q)^\perp$ and $\mathrm{wt}(c_\iota^+) = \mathrm{wt}(c)$. Codewords of this form are called *pull-backs*.

Proof. A j-space λ lies in $\mathrm{supp}(c_\iota^+)$ if and only if λ contains ι, and intersects π in a point of $\mathrm{supp}(c)$. Since every point of π lies in a unique j-space through ι, we get $\mathrm{wt}(c_\iota^+) = \mathrm{wt}(c)$. Now take a k-space κ. If $\iota \not\subset \kappa$, then κ contains no j-spaces of $\mathrm{supp}(c_\iota^+)$, hence $\kappa \cdot c_\iota^+ = 0$. If $\iota \subset \kappa$, then it easy to see that $\kappa \cdot c_\iota^+ = (\kappa \cap \pi) \cdot c = 0$. The last equality holds because κ intersects π in a $(k-j)$-space, and $c \in \mathcal{C}_{k-j}(n-j,q)^\perp$. $\square$

Remark 7.2. A codeword $c \in \mathcal{C}_{j,k}(n,q)$ is a pull-back if and only if all j-spaces of $\mathrm{supp}(c)$ go through the same $(j-1)$-space ι. If the latter holds, then $\daleth_\iota(c) \in \mathcal{C}_{k-j}(n-j,q)^\perp$, and $c = (\daleth_\iota(c))^+_\iota$.

The previous remark asserts that the standard words of $\mathcal{C}_{j,k}(n,q)^\perp$ (see Definition 3.5) are pull-backs if $j > 0$. In fact, they are pull-backs of standard words of $\mathcal{C}_{k-j}(n-j,q)^\perp$. Bagchi & Inamdar [BI02, Conjecture] conjectured that the minimum weight codewords of $\mathcal{C}_{j,k}(n,p)^\perp$ are standard words, for p prime. They proved it for $j = k-1$, see Result 3.6, and $q = 2$ [BI02, Proposition 3]. They also mention that it can be proven in the case $j = 0$, using the theory of [DGM70]. Lavrauw, Storme & Van de Voorde [LSVdV08, Theorem 12] gave a geometric proof for the case $j = 0$, using Result 3.6. We give a short, alternative proof. This requires the following result, which is a slight alteration of the original statement using Lemma 4.1 (2).

Result 7.3 ([AK92, Theorem 5.7.9])**.** *If p is prime, then $\mathcal{C}_k(n,p)^\perp = \mathcal{H}_{n-k}(n,p)$.*

Corollary 7.4. *If p is prime, the minimum weight codewords of $\mathcal{C}_k(n,p)^\perp$ are the scalar multiples of the standard words.*

Proof. A standard word of $\mathcal{C}_k(n,p)^\perp$ is the difference of two $(n-k)$-spaces through an $(n-k-1)$-space. This corollary now follows directly from Corollary 5.10 and Result 7.3. $\square$

Putting these considerations together simplifies the conjecture of Bagchi & Inamdar. To finish the proof of the conjecture, we need to show that minimum weight codewords of $\mathcal{C}_{j,k}(n,q)^\perp$, $j > 0$ and q prime, are pull-backs. It will turn out q need not even be prime.

Lemma 7.5. *If $j > 0$, then all codewords $c \in \mathcal{C}_{j,j+1}(n,q)^\perp$, with $\mathrm{wt}(c) < 2\theta_{n-j-1}$, are pull-backs. In particular, this applies to the minimum weight codewords.*

Proof. Take a non-zero codeword $c \in \mathcal{C}_{j,j+1}(n,q)^\perp$, with $\mathrm{wt}(c) < 2\theta_{n-j-1}$. Take a $(j-1)$-space ι, define $X := \{\lambda \in \mathrm{supp}(c) : \iota \subset \lambda\}$, and denote $x := |X|$. Assume that $X \neq \emptyset$.
Take a j-space $\lambda_1 \in X$. Then every other element λ_2 of X lies is a unique $(j+1)$-space through λ_1. Therefore, there are at least $\begin{bmatrix} n-j \\ (j+1)-j \end{bmatrix}_q - (x-1) = \theta_{n-j-1} - x + 1$ $(j+1)$-spaces κ through λ_1, not containing another element of X. Each such space κ contains another element λ_3 of $\mathrm{supp}(c) \setminus X$, otherwise $\kappa \cdot c = c(\lambda_1) \neq 0$, contradicting the fact that $c \in \mathcal{C}_{j,j+1}(n,q)^\perp$. Note that λ_3 doesn't lie in a $(j+1)$-space with another element $\lambda_2 \in X \setminus \{\lambda_1\}$. Otherwise, λ_2 would intersect λ_1 in ι and λ_3 in another $(j-1)$-space (since $\lambda_3 \notin X$), which implies that $\lambda_2 \subset \langle \lambda_1, \lambda_3 \rangle = \kappa$. This is in contradiction with the way we chose κ.
Thus, every $\lambda_1 \in X$ gives rise to at least $\theta_{n-j-1} - x + 1$ elements in $\mathrm{supp}(c) \setminus X$, none of which are counted twice. This yields

$$2\theta_{n-j-1} > \mathrm{wt}(c) \geqslant x(\theta_{n-j-1} - x + 1 + 1).$$

This leads to a contradiction for $x = 2$ and $x = \theta_{n-j-1}$. Since the above expression is quadratic in x, we can see that it must lead to a contradiction whenever $2 \leqslant x \leqslant \theta_{n-j-1}$.
Now take a j-space $\lambda_1 \in \mathrm{supp}(c)$ and a $(j+1)$-space κ through λ_1. As argued above, we know that κ must contain another j-space $\lambda_2 \in \mathrm{supp}(c)$. Then $\lambda_1 \cap \lambda_2$ must be some $(j-1)$-space ι. By the previous arguments, we know that there are at least $\theta_{n-j-1} + 1$ elements of $\mathrm{supp}(c)$ through ι. Assume that λ is an element of $\mathrm{supp}(c)$ not through ι. Then there is at most one $(j+1)$-space through λ containing ι. This means that there are at least $\theta_{n-j-1} - 1$ $(j+1)$-spaces through λ, all containing another element of $\mathrm{supp}(c)$ not through ι. This yields $\mathrm{wt}(c) \geqslant (\theta_{n-j-1} + 1) + 1 + (\theta_{n-j-1} - 1) > 2\theta_{n-j-1}$, a contradiction.
Therefore, all elements of $\mathrm{supp}(c)$ contain a common $(j-1)$-space ι. By Remark 7.2, this proves that c is a pull-back. This applies to the minimum weight codewords, since the minimum weight of $\mathcal{C}_{j,j+1}(n,q)$ is at most $2q^{n-j-1}$, see Result 3.6. $\square$

The previous lemma was an induction base for the main theorem of this section. Its proof requires the following construction.

Construction 7.6. [LSVdV08, Theorem 10] Take an n-space π in $\mathrm{PG}(n+m,q)$ and a codeword $c \in \mathcal{C}_{j,k}(\pi)^\perp$. Now define $c' \in V(j,n+m,q)$ as

$$c'(\lambda) := \begin{cases} c(\lambda) & \text{if } \lambda \subset \pi \\ 0 & \text{otherwise} \end{cases}.$$

Then $c' \in \mathcal{C}_{j,k+m}(n+m,q)$ and $\mathrm{wt}(c') = \mathrm{wt}(c)$. We call c' an *embedded codeword* or a *codeword embedded in an n-space.*

Proof. Take a $(k+m)$-space ρ in $\mathrm{PG}(n+m,q)$. Then ρ intersects π in a space of dimension at least k. As a consequence, we can write $\rho\cap\pi$ (as element of $V(j,\pi)$) as the sum of its k-dimensional subspaces. This yields

$$\rho\cdot c' = (\rho\cap\pi)\cdot c = \left(\sum_{\kappa\in G_k(\rho\cap\pi)} \kappa\right)\cdot c = \sum_{\kappa\in G_k(\rho\cap\pi)} (\kappa\cdot c) = 0.$$

Hence, $c' \in \mathcal{C}_{j,k+m}(n+m,q)^\perp$. It is trivial that $\mathrm{wt}(c') = \mathrm{wt}(c)$. □

Corollary 7.7.

$$d\left(\mathcal{C}_{j,k}(n,q)^\perp\right) \geqslant d\left(\mathcal{C}_{j,k+1}(n+1,q)^\perp\right).$$

Proof. Take a minimum weight codeword $c \in \mathcal{C}_{j,k}(n,q)^\perp$. Embedding it in some hyperplane of $\mathrm{PG}(n+1,q)$, yields a codeword of $\mathcal{C}_{j,k+1}(n+1,q)^\perp$ of equal weight. □

The proof of the next theorem was inspired by [LSVdV08, Section 4].

Theorem 7.8. *If $j>0$, then all minimum weight codewords of $\mathcal{C}_{j,k}(n,q)^\perp$ are pull-backs.*

Proof. Fix a value $j>0$. The theorem will be proved through induction on k. We already know it holds for $k=j+1$. Hence, assume that $k>j+1$, and that the theorem holds for $\mathcal{C}_{j,k-1}(n-1,q)^\perp$. Take a minimum weight codeword $c\in\mathcal{C}_{j,k}(n,q)^\perp$. We know that $\mathrm{wt}(c)\leqslant 2q^{n-k}$. Thus,

$$|\mathrm{supp}_0(c)| \leqslant \mathrm{wt}(c)\theta_j \leqslant 2q^{n-k}\theta_j.$$

Take a j-space $\lambda\in\mathrm{supp}(c)$. Assume that every $(j+1)$-space ρ through λ contains at least q^j points of $\mathrm{supp}_0(c)\setminus\lambda$. This yields that

$$|\mathrm{supp}_0(c)| \geqslant \begin{bmatrix} n-j \\ (j+1)-j \end{bmatrix}_q q^j + \theta_j = \theta_{n-j-1}q^j + \theta_j = \theta_{n-1} + q^j.$$

Putting these inequalities together implies that $2q^{n-k}\theta_j \geqslant \theta_{n-1}+q^j$, which leads to a contradiction, since $k\geqslant j+2$.
So take a $(j+1)$-space ρ through λ such that ρ contains less than q^j points of $\mathrm{supp}_0(c)\setminus\lambda$. In particular, this means that $\rho\nsubseteq\mathrm{supp}_0(c)$. Therefore, there exists a point $R\in\rho\setminus\mathrm{supp}_0(c)$. If $c\cdot\rho=0$, then ρ must contain at least one other j-space of $\mathrm{supp}(c)$ than λ, which would also mean that ρ contains at least q^j points of $\mathrm{supp}_0(c)\setminus\lambda$, a contradiction. Let π be a hyperplane not through R. We know from Lemma 5.2 (3, 4) that $c' := \mathrm{proj}^{(j)}_{R,\pi}(c)\in\mathcal{C}_{j,k-1}(n-1,q)^\perp$, and $\mathrm{wt}(c')\leqslant\mathrm{wt}(c)$. We also know that $c'(\rho\cap\pi) = c\cdot\rho\neq 0$, so $c'\neq\mathbf{0}$.
Because c is a minimum weight codeword, Corollary 7.7 shows that $\mathrm{wt}(c')=\mathrm{wt}(c)$ and that c' must be a minimum weight codeword as well. Since $\mathrm{wt}(c')=\mathrm{wt}(c)$, Lemma 5.2 (5) implies that no $(j+1)$-space through R contains more than one j-space of $\mathrm{supp}(c)$.

By the induction hypothesis, there exists a $(j-1)$-space $\iota \subset \pi$ contained in all j-spaces of $\mathrm{supp}(c')$. Now take a j-space $\lambda \in \mathrm{supp}(c)$. Then R projects λ onto a j-space through ι (note that this holds because λ is the only element of $\mathrm{supp}(c)$ in $\langle R, \lambda \rangle$, so it gets projected onto an element of $\mathrm{supp}(c)$). This means that $\langle R, \lambda \rangle$ contains $\rho_1 := \langle R, \iota \rangle$, hence λ intersects ρ_1 in a $(j-1)$-space.
Now look at how R was chosen. We took a $(j+1)$-space ρ through some $\lambda \in \mathrm{supp}(c)$, such that ρ contains less than q^j points of $\mathrm{supp}_0(c) \setminus \lambda$. Note that ρ_1 intersects ρ in at most a j-space, hence $\rho_1 \cup \lambda$ contains at most $2q^j + \theta_{j-1}$ points of ρ. Since ρ contains $\theta_{j+1} \geqslant 3q^j + \theta_{j-1}$ points, there exists a point $R_2 \in \rho \setminus (\rho_1 \cup \mathrm{supp}_0(c))$. Take a hyperplane π_2 not through R_2. Repeating the previous arguments yields again a j-space $\rho_2 = \langle R_2, \iota_2 \rangle$, for some $(j-1)$-space $\iota_2 \subset \pi_2$, such that every j-space of $\mathrm{supp}(c)$ intersects ρ_2 in $(j-1)$-space. Note that $R_2 \notin \rho_1$, so $\rho_1 \neq \rho_2$.
Now take a j-space $\lambda \in \mathrm{supp}(c)$. Then ρ_1 and ρ_2 both intersect λ in a $(j-1)$-space, hence $\dim(\rho_1 \cap \rho_2) \geqslant \dim(\rho_1 \cap \rho_2 \cap \lambda) \geqslant j-2$. Assume that $\dim(\rho_1 \cap \rho_2) = j-2$, then $\dim \langle \rho_1, \rho_2 \rangle = j+2$. Now every j-space $\lambda \in \mathrm{supp}(c)$ intersects ρ_1 and ρ_2 in a different $(j-1)$-space, thus $\lambda \subset \langle \rho_1, \rho_2 \rangle$. This means that c is the embedding of a codeword $c' \in \mathcal{C}_{j,k'}(j+2,q)^\perp$, with $(j+2)-k' = n-k$. This is only possible if $j < k' < j+2$, hence $k' = j+1$. Then c' is a pull-back by Lemma 7.5. Thus, c is a pull-back as well.
Now assume that $\dim(\rho_1 \cap \rho_2) = j-1$, and therefore $\dim \langle \rho_1, \rho_2 \rangle = j+1$. Furthermore, assume that there exists a j-space $\lambda \in \mathrm{supp}(c)$ not through $\rho_1 \cap \rho_2$. Then ρ_1 and ρ_2 intersect λ in distinct hyperplanes of λ, hence $\lambda \subset \langle \rho_1, \rho_2 \rangle$ and there exists a k-space κ that intersects $\langle \rho_1, \rho_2 \rangle$ in λ. Since every j-space of $\mathrm{supp}(c)$ either contains $\rho_1 \cap \rho_2$ or is contained in $\langle \rho_1, \rho_2 \rangle$, this means that λ is the only element of $\mathrm{supp}(c)$ contained κ. But then $c \cdot \kappa = c(\lambda) \neq 0$, contradicting the fact that $c \in \mathcal{C}_{j,k}(n,q)^\perp$. Thus, all j-spaces of $\mathrm{supp}(c)$ go through the $(j-1)$-space $\rho_1 \cap \rho_2$. By Remark 7.2, c is a pull-back. □

This reduces the minimum weight problem of $\mathcal{C}_{j,k}(n,q)^\perp$ to the case $j = 0$. The following result reduces it further to $k = 1$.

Result 7.9 ([LSVdV08, Theorem 11]). *Every minimum weight codeword of $\mathcal{C}_k(n,q)^\perp$ is embedded in an $(n-k+1)$-space.*

Theorem 7.8 can generalise some previous work on the codes $\mathcal{C}_{j,k}(n,q)^\perp$.

Corollary 7.10. *(1) $d\left(\mathcal{C}_{j,k}(n,q)^\perp\right) = d\left(\mathcal{C}_1(n-k+1,q)^\perp\right)$.*

(2) If p is prime, then the minimum weight codewords of $\mathcal{C}_{j,k}(n,p)^\perp$ are scalar multiples of the standard words, and thus have weight $2p^{n-k}$.

(3) If q is even, then $d\left(\mathcal{C}_{j,k}(n,q)^\perp\right) = (q+2)q^{n-k-1}$.

Proof. (1) This follows directly from Theorem 7.8 and Result 7.9.
(2) As noted previously, this follows from Corollary 7.4, Theorem 7.8, and the fact that a pull-back c_ι^+ is a standard word if and only if c is a standard word.
(3) This follows from Theorem 7.8 and Result 3.7. □

If q is odd and not prime, the minimum weight of $\mathcal{C}_1(n,q)^\perp$ remains an open problem. The best bounds known to the authors are the following.

Result 7.11 ([BI02, Theorem 3][LSVdV10, Corollary 4.15]). *If q is odd, then*

$$2q^{n-1} - 2\frac{q-p}{p}\theta_{n-2} \leqslant d\left(\mathcal{C}_1(n,q)^\perp\right) \leqslant 2q^{n-1} - \frac{q-p}{p-1}q^{n-2}.$$

It deserves be noted that the lower bound in the previous result was also obtained for $n = 2$ in [KMM09].

There are other interesting constructions. Small weight codewords of $\mathcal{C}_1(n,q)^\perp$ can be constructed from small weight codewords of $\mathcal{C}_1(2,q)^\perp$.

Construction 7.12. Let π be a plane in $\mathrm{PG}(n,q)$, and take $c \in \mathcal{C}_1(\pi)^\perp$. Let τ be an $(n-3)$-space, skew to π. Define $c_\tau^- \in V(0,n,q)$ as follows:

$$c_\tau^-(P) = \begin{cases} 0 & \text{if } P \in \tau, \\ c(\langle P, \tau\rangle \cap \pi) & \text{otherwise.} \end{cases}$$

Then $c_\tau^- \in \mathcal{C}_1(n,q)^\perp$ and $\mathrm{wt}(c_\tau^-) = \mathrm{wt}(c)q^{n-2}$.

This construction is also described in [BI02, Lemma 6]. Note that $\mathrm{supp}(c_\tau^-)$ is a truncated cone with base $\mathrm{supp}(c)$ and vertex τ.

In [DB12], subgeometries are used to construct small weight codewords. We can generalise this construction using field reduction. The idea is as follows (for more details see e.g. [LVdV15]). Choose an exponent $e > 1$. The projective space $\mathrm{PG}(n,q^e)$ can be recognised in $\mathrm{PG}(N,q)$ with $N = (n+1)e-1$. The points of $\mathrm{PG}(n,q^e)$ correspond to an $(e-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(N,q)$. In general, each k-space of $\mathrm{PG}(n,q^e)$ corresponds to a $((k+1)e-1)$-space $\mathcal{B}(\kappa)$ of $\mathrm{PG}(N,q)$, such that each element of $\mathcal{S}$ is either skew to $\mathcal{B}(\kappa)$ or completely contained in $\mathcal{B}(\kappa)$.

Construction 7.13. Let $e \in \mathbb{N}\backslash\{0,1\}$ and $N := (n+1)e-1$. Take a codeword $c \in \mathcal{C}_{2e-1}(N,q)^\perp$. Define

$$c' : G_0(n,q^e) \to \mathbb{F}_p : P \mapsto c \cdot \mathcal{B}(P).$$

Then $c' \in \mathcal{C}_1(n,q^e)^\perp$ and $\mathrm{wt}(c') \leqslant \mathrm{wt}(c)$.

Proof. Take a line l in $\mathrm{PG}(n,q^e)$. Then we know that $\{\mathcal{B}(P) : P \in l\}$ is a partition of the points of $\mathcal{B}(l)$. Therefore,

$$c' \cdot l = \sum_{P \in l} c'(P) = \sum_{P\in l} c \cdot \mathcal{B}(P) = \sum_{P' \in \cup_{P\in l}\mathcal{B}(P)} c(P') = c \cdot \mathcal{B}(l) = 0.$$

The last equality holds because $\mathcal{B}(l)$ is a $(2e-1)$-space in $\mathrm{PG}(N,q)$ and $c \in \mathcal{C}_{2e-1}(n,q)^\perp$. If a point P of $\mathrm{PG}(n,q^e)$ lies in $\mathrm{supp}(c')$, then $\mathcal{B}(P)$ must certainly contain a point of $\mathrm{supp}(c)$. Since the spread $\mathcal{S} := \{\mathcal{B}(P) : P \in G_0(n,q^e)\}$ partitions the points of $\mathrm{PG}(N,q)$, $\mathrm{supp}(c')$ cannot contain more points than $\mathrm{supp}(c)$. $\square$

Remark 7.14. If the codeword c in the above definition is a minimum weight codeword of $\mathcal{C}_{2e-1}(N,q)^\perp$, then it is embedded in an $((n-1)e+1)$-space π. In that case, it's not hard to check that $\mathrm{supp}(c')$ are the points P in $\mathrm{PG}(n,q^e)$, such that $\mathcal{B}(P)$ intersects π in a single point and this point belongs to $\mathrm{supp}(c)$.

8 The dimension

In general, the dimension of $\mathcal{C}_{j,k}(n,q)$ is still unknown. The dimension of $\mathcal{C}_k(k+1,q)$ has been determined independently in several articles.

Result 8.1 ([GD68, MM68, Smi69])**.**

$$\dim \mathcal{C}_k(k+1,q) = \binom{p+k}{k+1}^h + 1.$$

This formula has been generalised by Hamada to cover all codes $\mathcal{C}_k(n,q)$.

Result 8.2 ([Ham68])**.** *The dimension of $\mathcal{C}_k(n,q)$, with $q=p^h$, and p prime, is given by*

$$\dim \mathcal{C}_k(n,q) = \sum_{s_0,\dots,s_{h-1}} \prod_{j=0}^{h-1} \sum_{i=0}^{\left\lfloor \frac{s_{j+1}p-s_j}{p} \right\rfloor} (-1)^i \binom{n+1}{i}\binom{n+s_{j+1}p-s_j-ip}{n},$$

where $s_h = s_0$ and the summation runs over $s_0,\dots,s_{h-1}$ under the restriction that $k+1 \leqslant s_j \leqslant n+1$, and $0 \leqslant s_{j+1}p - s_j \leqslant (n+1)(p-1)$.

The following equality seems to have remained unnoticed.

Lemma 8.3.

$$\dim \mathcal{C}_{j,k}(n,q) = \dim \mathcal{C}_{n-k-1,n-j-1}(n,q).$$

Proof. As was noted in Subsection 3.1, $\mathcal{C}_{j,k}(n,q)$ can be seen as the row space of the p-ary incidence matrix of k-spaces and j-spaces of $\mathrm{PG}(n,q)$. Call this matrix A. Then by duality, A can also be seen as the transposed incidence matrix of $(n-j-1)$-spaces and $(n-k-1)$-spaces of $\mathrm{PG}(n,q)$. Thus, $\mathcal{C}_{n-k-1,n-j-1}(n,q)$ is the column space of A. Therefore, the dimensions of both codes equal the rank of A. ☐

Hence, the dimension of $\mathcal{C}_{j,k}(n,q)$ is known whenever $j=0$ or $k=n-1$. These are the only cases in which the dimension is known. As the expression in Result 8.2 is such a mouthful, one should not expect an easy formula for the general case to exist.

9 Open problems

A first open problem is solving the minimum weight problem of $\mathcal{C}_1(n,q)^\perp$. It would be interesting to investigate whether (all) minimum weight codewords of $\mathcal{C}_1(n,q)^\perp$, $n>2$, come from Construction 7.12, and it would be delightful if the answer is affirmative. In that case, the minimum weight problem is entirely reduced to $\mathcal{C}_1(2,q)^\perp$, which remains an interesting case in itself.

Secondly, it would also be nice if the characterisations for $\mathcal{C}_{j,k}(n,q)$ could be improved beyond the bound $W(j,k,q)$, and if the minimum weight of $\mathcal{H}_{j,k}(n,q)$ could be proven to be $2q^{k-j}\begin{bmatrix}k\\j\end{bmatrix}_q$ for small values of q as well.

Finally, determining a general formula for dim $\left(\mathcal{C}_{j,k}(n,q)\right)$ is an interesting challenge.

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